Probability Calculus 2021/2022
Problem set 12

1. Employment levels in different sewing factories in Bangladesh were compared. The number of needlewomen employed in particular units was a random variable with a uniform distribution over the interval $[100, 300]$. The wage of a needlewoman depends on the institution, and has an average of 2400 with a standard deviation of $400\sqrt{3}$. The correlation coefficient of the employment level and the wage level in the units compared was equal to 0.6. Find the best linear approximation of the relationship between wages and the employment level in a given factory.

2. We measure a physical value 100 times. The measurement errors are independent random variables with mean 0 and variance equal to 0.1. Using the Chebyshev-Bienaymé Inequality, find an upper bound to the probability that the absolute value of the total (aggregate) error exceeds 10.

3. Let X_1, X_2, ... be independent random variables with distributions

   (a) $\mathbb{P}(X_n = 0) = 1/2 = \mathbb{P}(X_n = 2)$

   (b) uniform over $[0, 1]$

   Does the sequence $(Y_n)_{n \geqslant 1}$, where $Y_n = X_1 \cdot X_2 \cdot \ldots \cdot X_n$, converge in probability? Almost surely? If yes, find the limits.

4. Let $X_1, X_2, \ldots$ be uncorrelated random variables, such that X_n has a uniform distribution over $[-1/n, 1/n]$. Does the sequence

   $$\frac{X_1 + X_2 + \ldots + X_n}{n}$$

   converge in probability?

5. Let $X_1, X_2, \ldots$ be independent random variables, such that $\mathbb{P}(X_n = n) = \mathbb{P}(X_n = -n) = \frac{1}{2}$. Does the sequence

   $$\frac{X_1 + X_2 + \ldots + X_n}{n}$$

   satisfy the WLLN? Converge in probability to 0?

6. Let $X_1, X_2, \ldots$ be independent random variables with exponential distribution with parameter 2. Does the sequence

   $$\frac{X_1 + X_2 + \ldots + X_n + 3}{2n + 31}, \qquad n = 1, 2, \ldots$$

   converge almost surely? What is the limit?

7. We randomly and independently draw points $A_1, A_2, \ldots$ from the interval $[0, 3]$. For a given n, let S_n denote the number of points among $A_1, A_2, \ldots, A_n$, which fall into the interval $[0, 1]$. Verify, whether $\frac{S_n}{n} \to \frac{1}{3}$ almost surely.

Some additional simple problems you should be able to solve on your own:

Theory (you should know going into this class)

1. Formulate the Chebyshev inequalities and the Bernstein inequality.
2. What does it mean that a sequence of random variables converges almost surely/in probability?
3. Provide the Weak and Strong Laws of Large Numbers for the Bernoulli Scheme.

Problems (you should know how to solve after this class)

A. A symmetric coin is tossed 100 times. Using the Bernstein inequality, assess the probability that heads will appear in at least 60% cases.

B. Let $X_1, X_2, \ldots$ be independent random variables from an exponential distribution with parameter 2. Verify, whether the sequence

$$\frac{X_1 + X_2 + \ldots + X_n + 3}{n + 31}, \qquad n = 1, 2, \ldots$$

converges in probability.

C. Let $X_1, X_2, \ldots$ be uncorrelated random variables, where X_n has a distribution given by $\mathbb{P}(X_n = -n) = \mathbb{P}(X_n = n) = 1/(2n^2)$, $\mathbb{P}(X_n = 0) = 1 - 1/n^2$ for $n \geqslant 1$. Verify, whether the sequence

$$\frac{X_1 + X_2 + \ldots + X_n}{n}, \qquad n = 1, 2, \ldots,$$

converges in probability.

D. Let $X_1, X_2, \ldots$ be independent random variables from a uniform distribution over the interval $[0, 1]$. Prove that the sequence

$$\frac{X_1 + X_2 + \ldots + X_n}{n}, \qquad n = 1, 2, \ldots,$$

converges in probability and find the limit.

E. Let $X_1, X_2, \ldots$ be independent random variables from a geometric distribution with parameter 0.1. Verify, whether the sequence

$$\frac{X_1 + X_2 + \ldots + X_n - 15}{n + 31}, \qquad n = 1, 2, \ldots$$

converges almost surely and if yes, find the limit.

F. Let $X_1, X_2, \ldots$ be independent random variables from a uniform distribution over the interval $[0, 1]$. Prove that the sequence

$$\frac{X_1 + X_2 + \ldots + X_n}{n}, \qquad n = 1, 2, \ldots,$$

converges almost surely and find the limit.

G. A transmitter sends signal X. A receiver picks up signal $Y = aX + Z$, where $a > 0$ is the amplification factor, and Z is interference. X and Z are independent random variables, such that $\mathbb{E}X = m$, $\operatorname{Var} X = 1$, $\mathbb{E}Z = 0$ and $\operatorname{Var} Z = \sigma^2$. Find the correlation coefficient of X and Y and the linear regression of X and Y.