Probability Calculus

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LAWS OF LARGE NUMBERS – APPLICATIONS. CENTRAL LIMIT THEOREM

Plan for Today

Laws of Large Numbers – examples of applications

- Central Limit Theorem
 - de Moivre-Laplace Theorem



Weak Laws of Large Numbers – reminder

1. Weak Law of Large Numbers for the Bernoulli Scheme

Let X_1, X_2, \ldots be independent with distributions $\mathbb{P}(X_n = 1) = p = 1 - \mathbb{P}(X_n = 0)$. We then have that (S_n/n) converges in probability to p; in other words, for any $\varepsilon > 0$, we have $\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{S_n}{n} - p\right| > \varepsilon\right) = 0$.

 $S_n = X_1 + X_2 + \ldots + X_n$



Weak Laws of Large Numbers – cont. – reminder

2. Weak Law of Large Numbers for uncorrelated random variables

Let X_1, X_2, \ldots be uncorrelated random variables with a common upper bound to their variances. Then, the sequence $(X_n)_{n \ge 1}$ satisfies the weak law of large numbers: $\frac{S_n - \mathbb{E}S_n}{n} \xrightarrow{\mathbb{P}} 0$, i.e. for any $\varepsilon > 0$ we have $\lim_{n \to \infty} \mathbb{P}\left(\left| \frac{S_n - \mathbb{E}S_n}{n} \right| > \varepsilon \right) = 0.$



Strong Laws of Large Numbers – reminder

1. Strong Law of Large Numbers for the Bernoulli Scheme

Let X_1, X_2, \ldots be a sequence of independent random variables, such that $\mathbb{P}(X_n = 1) = p = 1 - \mathbb{P}(X_n = 0), \quad n = 1, 2, \ldots$ Then, the sequence (S_n/n) converges almost surely to p; i.e., there exists an event Ω' of measure 1 such that for any $\omega \in \Omega'$, we have $\lim_{n \to \infty} \frac{S_n(\omega)}{n} = p.$



Strong Laws of Large Numbers – reminder cont.

2. Kolmogorov's Strong Law of Large Numbers

Let X_1, X_2, \ldots be a sequence of independent, identically distributed integrable random variables. Then, $\frac{S_n}{n} \xrightarrow[n \to \infty]{a.s.} \mathbb{E}X_1.$



Applications of the SLLN:

1. Convergence of the sample mean

$$\overline{X} = \frac{X_1 + X_2 + \ldots + X_n}{n} \xrightarrow[n \to \infty]{a.s.} \mathbb{E}X_1.$$

2. Convergence of the sample variance

$$S^{2} = \frac{1}{n} \sum_{k=1}^{n} (X_{k} - \overline{X})^{2} \xrightarrow[n \to \infty]{a.s.} \operatorname{Var} X_{1}.$$



Applications of the SLLN – cont.

3. Convergence of sample distributions: for

$$\mu_n(A) = \frac{1_A(X_1) + 1_A(X_2) + \ldots + 1_A(X_n)}{n}$$

we have $\mu_n(A) \xrightarrow[n \to \infty]{a.s.} \mathbb{E}1_A(X_1) = \mathbb{P}(X_1 \in A)$ 4. Convergence of sample CDFs: for $F_n(t) = \frac{1_{\{X_1 \leq t\}} + 1_{\{X_2 \leq t\}} + \ldots + 1_{\{X_n \leq t\}}}{n}$

we have $F_n(t) \xrightarrow[n \to \infty]{a.s.} F(t)$



Applications of SLLN – cont. (2)

5. Glivenko–Cantelli Theorem

Let X_1, X_2, \ldots be independent random variables from a distribution with a CDF F. Then, $\sup_{t \in \mathbb{R}} |F_n(t) - F(t)| \xrightarrow[n \to \infty]{a.s.} 0.$

- 6. Other examples
 - embarassing question

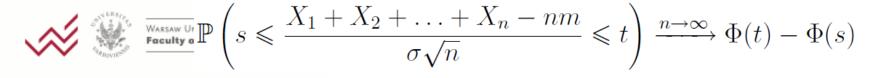


Central Limit Theorem

1. Classical version:

Let X_1, X_2, \ldots be identically distributed independent random variables, such that $\mathbb{E}X_1^2 < \infty$. If by $m = \mathbb{E}X_1$ we denote the mean, and by $\sigma^2 = \operatorname{Var} X_1$ the variance of this distribution, then for any $t \in \mathbb{R}$, we have that $\mathbb{P}\left(\frac{X_1 + X_2 + \dots + X_n - nm}{\sigma\sqrt{n}} \leqslant t\right) \xrightarrow{n \to \infty} \Phi(t),$ where $\Phi(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} \exp(-x^2/2) dx$ is the CDF of the standard normal distribution. also:

$$\mathbb{P}\left(s \leqslant \frac{X_1 + X_2 + \ldots + X_n - nm}{\sigma\sqrt{n}}\right) \xrightarrow[n \to \infty]{} 1 - \Phi(s)$$



De Moivre-Laplace Theorem

2. Theorem:

Let X_1, X_2, \ldots be a sequence of independent identically distributed random variables, such that $\mathbb{P}(X_n = 1) = p = 1 - \mathbb{P}(X_n = 0).$ Then, we have that for any s < t, $\mathbb{P}\left(s \leqslant \frac{X_1 + X_2 + \ldots + X_n - np}{\sqrt{np(1-p)}} \leqslant t\right) \xrightarrow{n \to \infty} \Phi(t) - \Phi(s).$

each inequality (both in the CLT and in dML) may be changed to strict without consequences



Central Limit Theorem

- 3. Examples
 - boys and girls
 - how many students should be accepted?
 - aggregate errors
 - confidence intervals



