Probability Calculus

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Laws of Large Numbers – Applications.

Central Limit Theorem

Plan for Today

- Laws of Large Numbers – examples of applications
- Central Limit Theorem
  - de Moivre-Laplace Theorem

Weak Laws of Large Numbers – reminder

1. Weak Law of Large Numbers for the Bernoulli Scheme

Let $X_1, X_2, \ldots$ *be independent with distributions* $\mathbb{P}(X_n = 1) = p = 1 - \mathbb{P}(X_n = 0)$. *We then have that* (S_n/n) *converges in probability to* p*; in other words, for any* $\varepsilon > 0$, *we have* $\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{S_n}{n} - p\right| > \varepsilon\right) = 0$.

$$S_n = X_1 + X_2 + \ldots + X_n$$

Weak Laws of Large Numbers – cont. – reminder

2. Weak Law of Large Numbers for uncorrelated random variables

Let $X_1, X_2, \ldots$ *be uncorrelated random variables with a common upper bound to their variances. Then, the sequence* $(X_n)_{n\geqslant 1}$ *satisfies the weak law of large numbers:* $\frac{S_n - \mathbb{E}S_n}{n} \xrightarrow{\mathbb{P}} 0$, *i.e. for any* $\varepsilon > 0$ *we have* $\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{S_n - \mathbb{E}S_n}{n}\right| > \varepsilon\right) = 0.$

Strong Laws of Large Numbers – reminder

1. Strong Law of Large Numbers for the Bernoulli Scheme

Let $X_1, X_2, \ldots$ *be a sequence of independent random variables, such that*
$\mathbb{P}(X_n = 1) = p = 1 - \mathbb{P}(X_n = 0), \quad n = 1, 2, \ldots.$
Then, the sequence (S_n/n) *converges almost surely to* p*; i.e., there exists an event* Ω' *of measure 1 such that for any* $\omega \in \Omega'$*, we have*
$\lim_{n\to\infty} \frac{S_n(\omega)}{n} = p.$

Strong Laws of Large Numbers – reminder cont.

2. Kolmogorov's Strong Law of Large Numbers

Let $X_1, X_2, \ldots$ *be a sequence of independent, identically distributed integrable random variables. Then,*

$$\frac{S_n}{n} \xrightarrow[n\to\infty]{a.s.} \mathbb{E}X_1.$$

Applications of the SLLN:

1. Convergence of the sample mean

$$\overline{X} = \frac{X_1 + X_2 + \ldots + X_n}{n} \xrightarrow[n\to\infty]{a.s.} \mathbb{E}X_1.$$

2. Convergence of the sample variance

$$S^2 = \frac{1}{n}\sum_{k=1}^{n}(X_k - \overline{X})^2 \xrightarrow[n\to\infty]{a.s.} \mathrm{Var}X_1.$$

Applications of the SLLN – cont.

3. Convergence of sample distributions: for

$$\mu_n(A) = \frac{1_A(X_1) + 1_A(X_2) + \ldots + 1_A(X_n)}{n}$$

we have $\mu_n(A) \xrightarrow[n\to\infty]{a.s.} \mathbb{E}1_A(X_1) = \mathbb{P}(X_1 \in A)$

4. Convergence of sample CDFs: for

$$F_n(t) = \frac{1_{\{X_1 \leqslant t\}} + 1_{\{X_2 \leqslant t\}} + \ldots + 1_{\{X_n \leqslant t\}}}{n}$$

we have $F_n(t) \xrightarrow[n\to\infty]{a.s.} F(t)$

Applications of SLLN – cont. (2)

5. Glivenko–Cantelli Theorem

Let $X_1, X_2, \ldots$ be independent random variables from a distribution with a CDF F. Then,

$$\sup_{t\in\mathbb{R}} |F_n(t) - F(t)| \xrightarrow[n\to\infty]{a.s.} 0.$$

6. Other examples

- embarassing question

Central Limit Theorem

1. Classical version:

Let $X_1, X_2, \ldots$ *be identically distributed independent random variables, such that* $\mathbb{E}X_1^2 < \infty$. *If by* $m = \mathbb{E}X_1$ *we denote the mean, and by* $\sigma^2 = \mathrm{Var}X_1$ *the variance of this distribution, then for any* $t \in \mathbb{R}$, *we have that*

$$\mathbb{P}\left(\frac{X_1+X_2+\ldots+X_n-nm}{\sigma\sqrt{n}} \leqslant t\right) \xrightarrow{n\to\infty} \Phi(t),$$

where $\Phi(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)dx$ *is the CDF of the standard normal distribution.*

also:

$$\mathbb{P}\left(s \leqslant \frac{X_1 + X_2 + \ldots + X_n - nm}{\sigma\sqrt{n}}\right) \xrightarrow{n\to\infty} 1 - \Phi(s)$$

$$\mathbb{P}\left(s \leqslant \frac{X_1 + X_2 + \ldots + X_n - nm}{\sigma\sqrt{n}} \leqslant t\right) \xrightarrow{n\to\infty} \Phi(t) - \Phi(s)$$

De Moivre-Laplace Theorem

2. Theorem:

Let $X_1, X_2, \ldots$ *be a sequence of independent identically distributed random variables, such that* $\mathbb{P}(X_n = 1) = p = 1 - \mathbb{P}(X_n = 0)$.
Then, we have that for any $s < t$,

$$\mathbb{P}\left(s \leqslant \frac{X_1+X_2+\ldots+X_n-np}{\sqrt{np(1-p)}} \leqslant t\right) \xrightarrow{n\to\infty} \Phi(t) - \Phi(s).$$

each inequality (both in the CLT and in dML) may be changed to strict without consequences

Central Limit Theorem

3. Examples

- boys and girls
- how many students should be accepted?
- aggregate errors
- confidence intervals