Probability Calculus 2021/2022
Problem set 11

1. We roll a die twice. Let X and Y denote the numbers obtained in the first and second roll, respectively. Calculate $\mathbb{E}(Y|X)$, $\mathbb{E}(X+Y|X)$ and $\mathbb{E}(X|X+Y)$.

2. From the set $\{1, 2, \ldots, 10\}$ we randomly draw, without replacement, two numbers. Let X be the smaller and Y the larger of the two values. Calculate $\mathbb{E}(X|Y)$ and $\mathbb{E}(XY+X|X)$.

3. The monthly energy usage in a plant has a uniform distribution over $[200, 250]$. For a given usage level λ, the amount of CO_2 emissions has an exponential distribution with parameter $5 - \lambda/100$. Find the (unconditional) density of the level of emissions.

4. Let (X, Y) be a random vector with a uniform distribution over a triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$. Calculate $\mathbb{E}(Y|X)$, $\mathbb{E}(XY^2 + 3X^2Y - 1|X)$ and $\mathbb{P}(Y \leqslant \frac{1}{2}|X)$.

5. We roll a die once, and then again as many times as there were points during the first roll. Let X denote the total sum of points obtained during the experiment (including the first roll). Find $\mathbb{E}X$.

Some additional simple problems you should be able to solve on your own:

Theory (you should know going into this class)

1. What is the definition of a conditional expectation for a discrete random variable?
2. What is a conditional density function? What is the definition of a conditional expectation for a continuous random variable?

Problems (you should know how to solve after this class)

3. Knowing that $\mathbb{P}(Y = 1|X = 5) = 1/3$ and $\mathbb{P}(Y = 5|X = 5) = 2/3$, find $\mathbb{E}(Y|X = 5)$ and $\mathbb{E}(XY^2|X = 5)$.
4. There are two white balls, with numbers 1 and 2, and three black balls, with numbers 1, 2 and 3, in a box. Two balls were drawn from the box without replacement. Let X denote the maximum number obtained, and Y denote the number of white balls drawn. Find $\mathbb{E}(Y|X)$ and $\mathbb{E}(X|Y)$.
5. A coin was tossed three times. Let X denote the number of heads and

$$Y = \begin{cases} 1 & \text{if the last toss was heads,} \\ 0 & \text{otherwise.} \end{cases}$$

   Find $\mathbb{E}(X|Y)$ and $\mathbb{E}(XY|X)$.

6. Let (X, Y) be a random vector from a uniform distribution over a triangle with vertices $(2, 0)$, $(0, 1)$ and $(-1, 0)$. Calculate $\mathbb{E}(X|Y)$ and $\mathbb{E}(X^2 + XY|Y)$.
7. Let (X, Y) be a random vector with density

$$g(x, y) = (x + y)1_{\{0 \leqslant x \leqslant 1,\, 0 \leqslant y \leqslant 1\}}.$$

   Find $\mathbb{E}(X|Y)$ and $\mathbb{E}(\sin X + Y|Y)$.