Probability Calculus 2021/2022
Problem set 10

1. We roll a die until we obtain an even number. Let X denote the number of rolls, and Y the number obtained in the last roll.
   (a) Find the distribution of the vector (X, Y).
   (b) Calculate $\mathrm{Cov}(X, Y)$. Are X and Y independent?
2. From a deck of 52 cards we draw 5 cards a) with replacement, b) without replacement. Let X denote the number of clubs among the drawn cards. Calculate the mean and the variance of X.
3. Let (X, Y) have a uniform distribution over the square
   $$S = \{(x, y) \in \mathbb{R}^2 : |x| + |y| \leqslant 1\}.$$
   (a) Find the marginal densities of X and Y
   (b) Calculate $\mathrm{Cov}(X, Y)$. Are X and Y independent?
4. Let X and Y be independent random variables, such that X has an exponential distribution with parameter 1, and Y has a distribution with density
   $$g_Y(y) = ye^{-y}1_{[0,\infty)}(y).$$
   Find the probability density of variable $X + Y$.
5. Let (X, Y) be a normal random vector with mean $(0, 0)$ and a covariance matrix
   $$\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}.$$
   (a) Are X and Y independent?
   (b) Find the density of the vector (X, Y).
   (c) What is the distribution of the variable $X + 2Y + 1$?
   (d) For which value of a, are X and $X + aY$ independent?

Some additional simple problems:

Theory(you should know going into this class)

1. What does it mean that random variables are uncorrelated? How does that relate to independence?

2. Provide the formula for the variance of a sum of random variables.

Problems (you should know how to solve after this class)

3. X, Y and Z are random variables with identical distributions, such that $\mathrm{Var}(X+Y+Z) = 21$, $\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,Z) = \mathrm{Cov}(Z,X) = 1$. Find $\mathrm{Var}X$ and $\mathrm{Var}(X+Y)$.

4. Let (X,Y) be a random vector with density
$$g(x,y) = \frac{1}{2\pi} \exp\left(-\frac{2x^2 - 2xy + y^2}{2}\right).$$
Find the covariance matrix of (X,Y), the distribution of the random vector $2X - Y + 2$ and verify whether X and $X - Y$ are independent.

5. Let X and Y be independent random variables with uniform distributions over intervals $[0,1]$ and $[0,2]$, respectively. Find the density function of the variable $X+Y$.

6. From a $[0,2] \times [0,2]$ square we randomly and independently draw 20 points. Let X denote the number of points from among those drawn, that fall into the unit square $[0,1] \times [0,1]$. Calculate the expected value and the variance of X.