

Probability Calculus 2021/2022
Problem set 9

1. A die is rolled twice. Let X denote the number of sixes obtained, and Y – the number of ones.
 - a) Find the support of (X, Y)
 - b) Calculate the covariance of X and Y
 - c) Find the probability that $X > Y$.
2. A random vector (X, Y) has a distribution given by

$$\mathbb{P}\left((X, Y) = (k, l)\right) = \frac{4kl}{n^2(n+1)^2}, \quad k, l = 1, 2, \dots, n.$$

- a) Find $\mathbb{P}(X + Y = n + 1)$.
 - b) Find the marginal distributions of X and Y .
 - c) Calculate $\text{Cov}(X, Y)$.
3. A random vector (X, Y) has a density $g(x, y) = Cx1_{\{0 \leq x \leq 1, 0 \leq y \leq 1\}}$.
 - a) Calculate C
 - b) Calculate $\mathbb{P}(X + Y < 1)$ and $\mathbb{P}(Y \leq 1/2)$.
 - c) Find the marginal distributions of X and Y .
 - e) Calculate $\text{Cov}(X, Y)$.
4. The covariance matrix of random vector (X, Y) is equal to

$$\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}.$$

Calculate the correlation coefficient of $X + 3Y, 2X - Y$.

5. Let (X, Y) be a random vector with density

$$g(x, y) = xe^{-y}1_{\{0 \leq x \leq y\}}.$$

- a) Find the marginal distribution of X .
 - b) Calculate $\mathbb{E}e^{Y/2}$.
 - c) Calculate the CDF of (X, Y) at $(1, 1)$.
 - d) Find the distribution of the variable Y/X .

Some additional problems

Theory (you should know going into this class)

1. What is the CDF of a two-dimensional random vector (X, Y) ? What are the marginal distributions?
2. Define the covariance and the correlation coefficient of variables X, Y .
3. Define the covariance matrix of variable (X, Y) .

Problems (you should know how to solve after this class)

4. A symmetric coin was tossed three times. Let X denote the number of heads in the last toss, and Y - the overall number of heads. Find $\mathbb{P}(X = Y)$ and $\text{Cov}(X, Y)$.
5. Let (X, Y) be a random vector with density $g(x, y) = Cxy1_{\{(x,y):0 \leq x \leq y \leq 1\}}$. Find C and $\mathbb{P}(X \geq 1/2)$.
6. Let (X, Y) be a random vector with density $g(x, y) = 1_{\{0 \leq y \leq 1 - |x|\}}$.
 - a) Find the CDF of (X, Y) at point $(1, \frac{1}{2})$.
 - b) Find the (marginal) densities of X and Y .
 - c) Calculate $\text{Cov}(X, Y)$.
 - d) Find the distribution of $Y - X$.
7. We roll a regular die until a six appears. Let X denote the overall number of tosses, and Y - the number of fives.
 - a) Find the distribution of (X, Y) .