# Probability Calculus


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**INDEPENDENCE OF RV**

# Plan for Today

1. Expected value and covarance matrix of a RV
2. Independence of random variables
3. Multidimensional Normal RV

# Expected value and covariance matrix

Definitions:

*Let* $(X,Y)$ *be a two-dimensional random vector.*
*Then, we have:*
*(i) If* $X$ *and* $Y$ *have expected values, then the* **expected value** $\mathbb{E}(X,Y)$ *of the vector* $(X,Y)$ *is the vector* $(\mathbb{E}X,\mathbb{E}Y)$.
*(ii) If* $X$ *and* $Y$ *have variances, then the* **covariance matrix** *of the vector* $(X,Y)$ *is the matrix*

$$\begin{bmatrix} \mathrm{Var}X & \mathrm{Cov}(X,Y) \\ \mathrm{Cov}(X,Y) & \mathrm{Var}Y \end{bmatrix}$$

*For higher dimensions* ($\mathbb{R}^d$, $d \geqslant 3$), *we have, similarly:*
*the expected value is the vector* $(\mathbb{E}X_1, \mathbb{E}X_2, \ldots, \mathbb{E}X_d)$,
*and the covariance matrix is the matrix* $(\mathrm{Cov}(X_i, X_j))_{1 \leqslant i,j \leqslant d}$.

# Properties of EX and the covariance matrix

*Let* $X = (X_1, X_2, \ldots, X_n)$ *be a random vector of dimension* $n$, *and* $A$ – *a* $m \times n$ *matrix.*
*(i) If* $X$ *has a finite expected value, then* $AX$ *also has a finite expected value, and* $\mathbb{E}(AX) = A\mathbb{E}X$.
*(ii) If the covariance matrix* $Q_X$ *of the vector* $X$ *exists, then there exists also the covariance matrix of the vector* $AX$, *and it is equal to* $Q_{AX} = AQ_XA^t$.

# Independent RV

## 1. Definition of independence

*Variables* $X_1, \ldots, X_n : \Omega \to \mathbb{R}$ *are* **independent**, *if for any sequence of Borel sets* $B_1, B_2, \ldots, B_n$, *we have*

$$\mathbb{P}(X_1 \in B_1, X_2 \in B_2, \ldots, X_n \in B_n)$$
$$= \mathbb{P}(X_1 \in B_1) \cdot \mathbb{P}(X_2 \in B_2) \cdot \ldots \cdot \mathbb{P}(X_n \in B_n).$$

## 2. Independence of discrete RV

*Let* $X_1, X_2, \ldots, X_n$ *be discrete random variables with supports* $S_{X_i}$, *respectively. In this case,* $X_1, X_2, \ldots, X_n$ *are independent if and only if for any sequence* $x_1, x_2, \ldots, x_n$ *such that* $x_i \in S_{X_i}$, $i = 1, 2, \ldots, n$, *we have*

$$\mathbb{P}(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$$
$$= \mathbb{P}(X_1 = x_1) \cdot \mathbb{P}(X_2 = x_2) \cdot \ldots \cdot \mathbb{P}(X_n = x_n).$$

# Independent RV – cont.

3. Example
4. Independence of continuous RV

*Let* $X_1, X_2, \ldots, X_n : \Omega \to \mathbb{R}$ *be continuous random variables with probability densities* $g_1, g_2, \ldots, g_n$*, respectively. In this case,* $X_1, X_2, \ldots, X_n$ *are independent if and only if* $g : \mathbb{R}^n \to [0, \infty)$*, defined as*

$$g(x_1, x_2, \ldots, x_n) = g_1(x_1) \cdot g_2(x_2) \cdot \ldots \cdot g_n(x_n),$$

*is a density function of the distribution* $\mu_{(X_1, X_2, \ldots, X_n)}$.

5. Examples
   - uniform distribution on square
   - uniform distribution on circle

# Independent RV – cont. (2)

## 6. Transformations of RV

*Let* $X_{1,1}, X_{1,2}, \ldots, X_{1,k_1}, X_{2,1}, X_{2,2}, \ldots, X_{2,k_2}, \ldots, X_{n,1}, X_{n,2}, \ldots, X_{n,k_n}$ *be independent random variables, and* $\varphi_i : \mathbb{R}^{k_i} \to \mathbb{R}$, $i = 1, 2, \ldots, n$ *be Borel functions. We then have that the variables*

$$Y_1 = \varphi_1(X_{1,1}, X_{1,2}, \ldots, X_{1,k_1}),$$

$$Y_2 = \varphi_2(X_{2,1}, X_{2,2}, \ldots, X_{2,k_2}),$$

$$\ldots$$

$$Y_n = \varphi_n(X_{n,1}, X_{n,2}, \ldots, X_{n,k_n})$$

*are independent.*

# Properties of independent RV

## 2. Expected value of product

*Let* $X_1, X_2, \ldots, X_n$ *be independent random variables with expected values. Then, the variable* $X = X_1 \cdot X_2 \cdot \ldots \cdot X_n$ *also has an expected value, and we have*

$$\mathbb{E}X = \mathbb{E}(X_1 \cdot X_2 \cdot \ldots \cdot X_n) = \mathbb{E}X_1 \cdot \mathbb{E}X_2 \cdot \ldots \cdot \mathbb{E}X_n.$$

## 3. Example

## 4. Covariance of independent RV

*Let* $X$ *and* $Y$ *be independent random variables, such that* $\mathbb{E}|XY| < \infty$. *We then have* $\mathrm{Cov}(X, Y) = 0$.

## 5. Non-correlation

# Properties of independent RV – cont.

6. One-way implication only!

independence $\Rightarrow$ non-correlation but

$\Leftarrow$ IS NOT TRUE!

7. Example – uniform distribution on circle

8. Sum of variances

*Let $X_1, X_2, \ldots, X_n$ be independent random variables with finite variances. Then, the variable $X = X_1 + X_2 + \ldots + X_n$ also has a finite variance, and we have*

$$\mathrm{Var}X = \mathrm{Var}(X_1 + X_2 + \ldots + X_n)$$
$$= \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \ldots + \mathrm{Var}(X_n).$$

# Properties of independent RV – cont.

6. One-way implication only!

independence $\Rightarrow$ non-correlation but

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8. Sum of variances

*Let* $X_1, X_2, \ldots, X_n$ *be* ~~*independent*~~ *random variables with finite variances. Then, the variable* $X = X_1 + X_2 + \ldots + X_n$ *also has a finite variance, and we have*

$$\mathrm{Var}X = \mathrm{Var}(X_1 + X_2 + \ldots + X_n)$$
$$= \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \ldots + \mathrm{Var}(X_n).$$
$$+ 2\sum_{i<j} \mathrm{Cov}(X_i, X_j).$$

# Properties of independent RV – cont. (2)

9. Example – sum of points on dice

10. Convolution of density functions

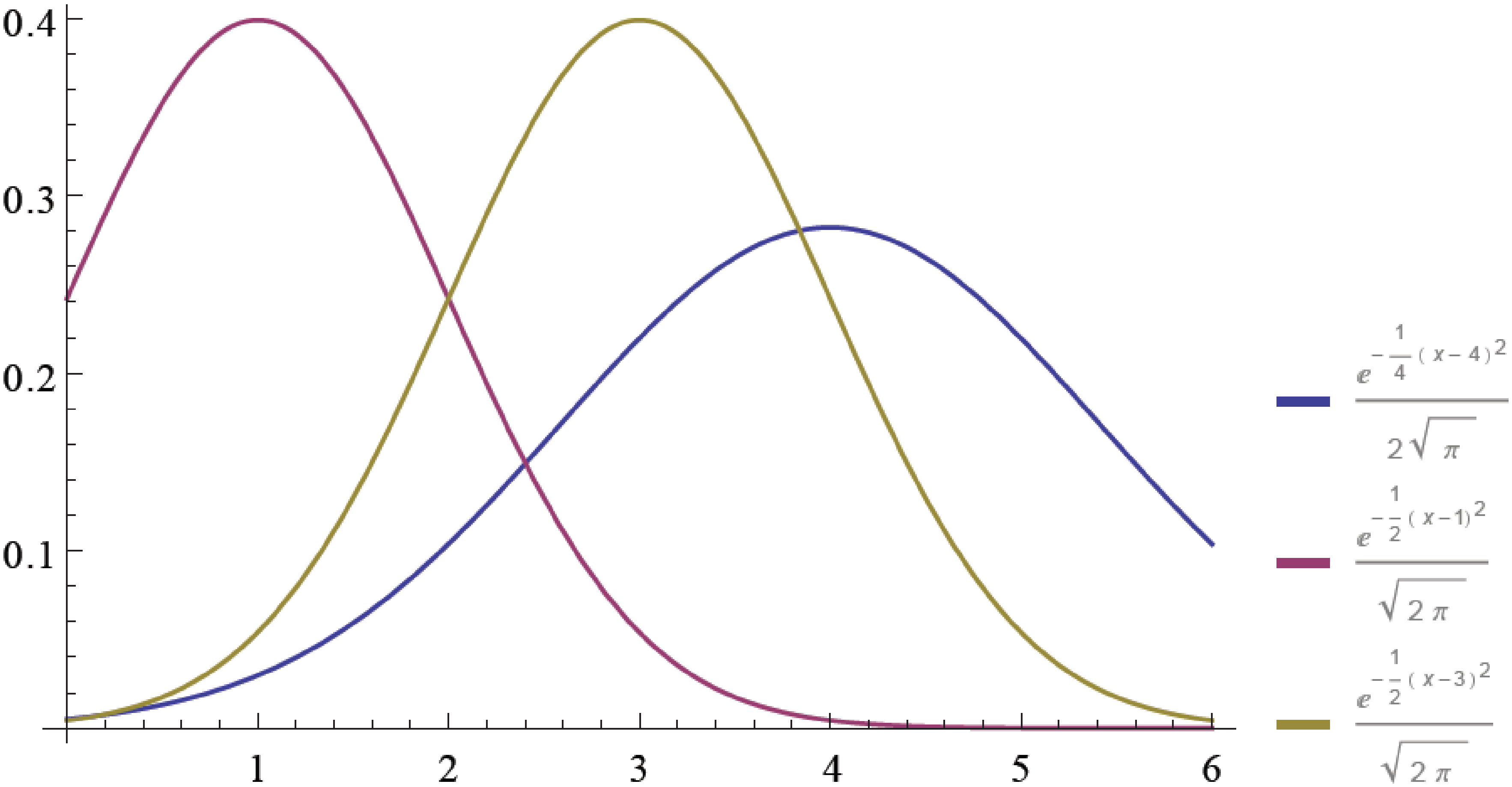
*Let $X$ and $Y$ be independent random variables with densities $g_X$ and $g_Y$, respectively. Then, the density of the variable $X+Y$ may be presented as a* **convolution** *of densities $g_X$ and $g_Y$:*

$$g_{X+Y}(t) = g_X * g_Y(t)$$

$$= \int_{\mathbb{R}} g_X(x) g_Y(t-x) dx = \int_{\mathbb{R}} g_X(t-y) g_Y(y) dy$$

11. Example

# Convolution of densities – example



# Multidimensional Normal RV

1. Definition

*Let* $m = (m_1, m_2, \ldots, m_n)$ *be a vector in* $\mathbb{R}^n$ *and let* $A$ *be a positive definite* $n \times n$ *matrix (i.e. such that* $x^t Ax > 0$ *for any nonzero vector* $x \in \mathbb{R}^n$*). A distribution with density*

$$g(x) = \frac{\sqrt{\det A}}{(2\pi)^{n/2}} \exp\left(-\frac{(x-m)^t A(x-m)}{2}\right), \qquad x \in \mathbb{R}^n$$

*is a* **normal** *distribution with mean* $m$ *and a covariance matrix* $Q = A^{-1}$.

2. Affine transformations of normal RV

# Two-dimensional normal RV

3. Two-dimensional normal RV with mean $m = (m_1, m_2)$ and a covariance matrix Q

$$g(x,y) = \frac{\sqrt{a_{11}a_{22} - a_{12}^2}}{2\pi}$$

$$\cdot \exp\left(-\frac{1}{2}(a_{11}(x-m_1)^2 + 2a_{12}(x-m_1)(y-m_2) + a_{22}(y-m_2)^2)\right)$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} = Q^{-1}$$

# Two-dimensional normal RV

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$$\sqrt{\det A}$$

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$$\cdot \exp\left(-\frac{1}{2}\left(a_{11}(x-m_1)^2 + 2a_{12}(x-m_1)(y-m_2) + a_{22}(y-m_2)^2\right)\right)$$

$$(x-m)^t A (x-m)$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} = Q^{-1}$$

# Condition of independence of normal RV

## 4. Theorem

*Let* $X = (X_1, X_2, \ldots, X_n)$ *be a normal variable, and let* $X_1, X_2, \ldots, X_n$ *be uncorrelated. Then,* $X_1, X_2, \ldots, X_n$ *are independent.*