Probability Calculus

Anna Janicka

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VARIANCE

MOMENTS

EMPIRICAL DISTRIBUTIONS

Plan for Today

1. Calculating EX – examples cont.
2. Variance
3. Moments
4. Empirical distributions
5. Intro to random vectors

Expected value – cont

Calculating EX based on the CDF – reminder

Let X be a non-negative random variable.
(i) If $\int_0^\infty \mathbb{P}(X > t)dt < \infty$, then X has an expected value and $\mathbb{E}X = \int_0^\infty \mathbb{P}(X > t)dt$.
(ii) If $p \in (0, \infty)$ and $\int_0^\infty pt^{p-1}\mathbb{P}(X > t)dt < \infty$, then X^p has an expected value and
$\mathbb{E}X^p = \int_0^\infty pt^{p-1}\mathbb{P}(X > t)dt$.

Examples

- geometric distribution
- exponential distribution
- *p*-th moments
- non-discrete non-continuous RV

Variance

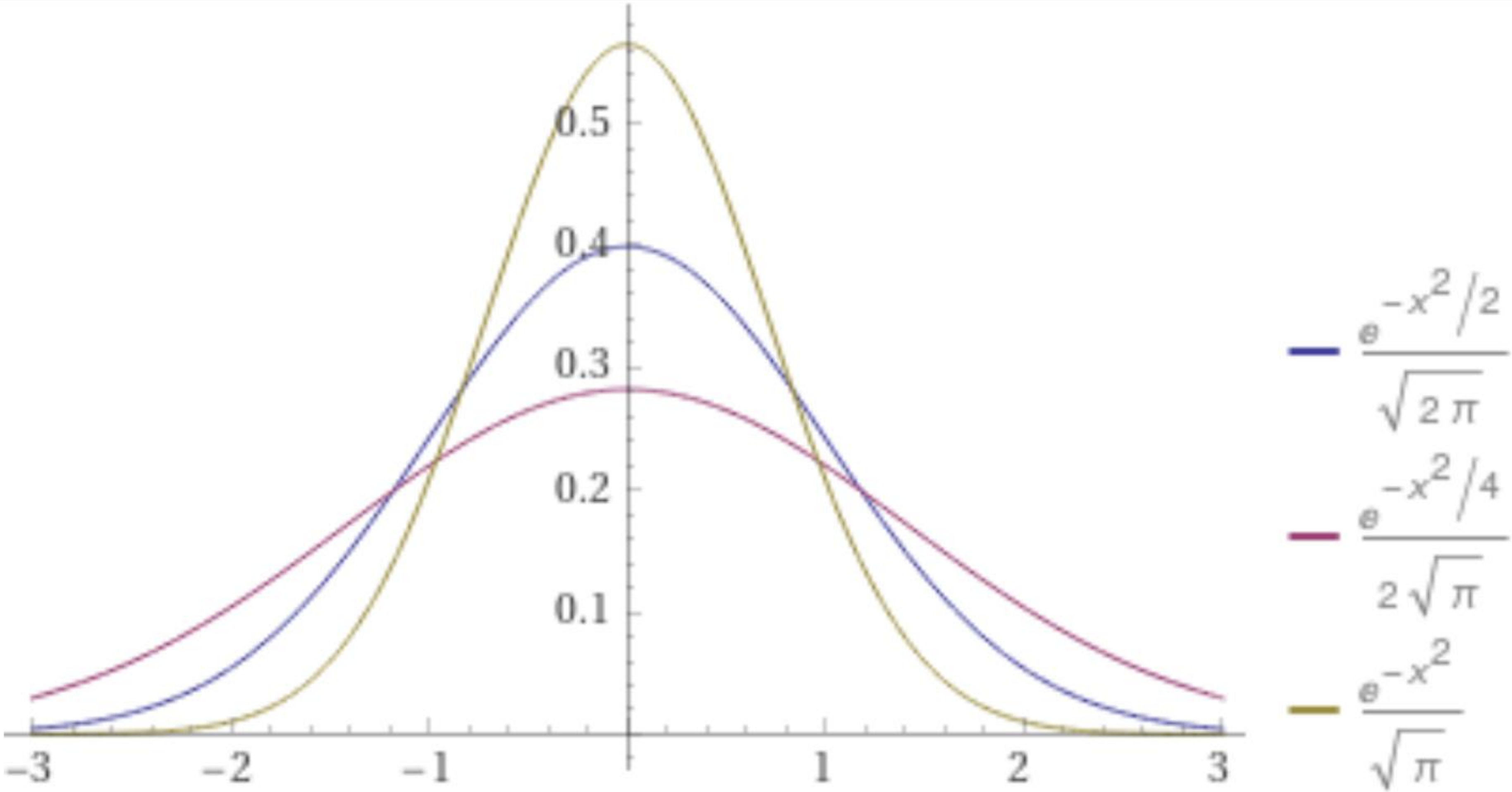
1. Definition

Let X be a random variable such that $\mathbb{E}|X| < \infty$ *and* $\mathbb{E}(X - \mathbb{E}X)^2 < \infty$. *The* **variance** *of X is defined as* $D^2X = VarX = \mathbb{E}(X - \mathbb{E}X)^2$.
The **standard deviation** *of variable X is the square root of the variance:* $\sigma_X = \sqrt{D^2X}$.

2. Properties

- depends on distribution only
- exists if single condition on EX^2, if limited
- simplified calculations: $D^2X = \mathbb{E}X^2 - (\mathbb{E}X)^2$
- interpretation

Variance – interpretation



Variance – cont.

3. Examples:
   - interpretation
   - die roll
   - uniform distribution

4. Properties, theorem:

Let X be a random variable with a variance.
(i) $D^2X \geqslant 0$, *and the equality holds if and only if there exists a value* $a \in \mathbb{R}$ *such that* $\mathbb{P}(X = a) = 1$.
(ii) $D^2(bX) = b^2 D^2 X$ *for any* $b \in \mathbb{R}$.
(iii) $D^2(X + c) = D^2 X$ *for any* $c \in \mathbb{R}$.

Variance – cont. (2)

5. Parameters of the normal distribution:
$N(m, \sigma^2)$

variance

mean

Moments

1. Definitions

For $p \in (0, \infty)$, *we define:*

(i) the **absolute moment** *of rank p for random variable* X *as* $\mathbb{E}|X|^p$ *(if this value is finite);*

For $p \in \mathbb{N}$, *we define:*

(ii) the **moment** *of rank p for random variable* X *as* $\mathbb{E}X^p$ *(provided that the p-th absolute moment exists);*

(iii) the **central moment** *of rank p for random variable* X *as* $\mathbb{E}(X - \mathbb{E}X)^p$ *(provided that the p-th absolute moment exists).*

Moments: skewness, kurtosis

2. Definitions

Let X *be a random variable such that* $\mathbb{E}|X|^3 < \infty$.
The **skewness** *of* X *is*

$$\alpha_3 = \frac{\mathbb{E}(X-\mathbb{E}X)^3}{(D^2X)^{3/2}} = \frac{\mathbb{E}(X-\mathbb{E}X)^3}{\sigma_X^3}.$$

Let X *be a random variable such that* $\mathbb{E}|X|^4 < \infty$.
The **kurtosis** *of* X *is*

$$\alpha_4 = \frac{\mathbb{E}(X-\mathbb{E}X)^4}{(D^2X)^2} - 3 = \frac{\mathbb{E}(X-\mathbb{E}X)^4}{\sigma_X^4} - 3.$$

3. Example: standard normal distribution

Empirical distributions

1. In reality, we frequently do not know the distributions of random variables, and work with *samples* instead.
2. *Let* $X_1, X_2, \ldots, X_n$ *be random variables with unknown distributions. An* **Empirical distribution (measure)** *for this sample is* $\mu_n(A) = \frac{1}{n}\sum_{i=1}^{n}\delta_{X_i}(A) = \frac{|\{i \leqslant n :\, X_i \in A\}|}{n}$,

Empirical distributions – cont.

3. *An* **empirical distribution function** *of the sample* $X_1, X_2, \ldots, X_n$ *is the function* $F\colon \mathbb{R} \to [0,1]$, *such that* $F_n(t) = \mu_n((-\infty, t]) = \frac{|\{i \leqslant n\colon X_i \leqslant t\}|}{n}$.

this is the CDF of the empirical distribution

4. *A* **Quantile** *of rank* p *of the sample* $X_1, \ldots, X_n$ *is any number* x_p, *such that*
$\mu_n((-\infty, x_p]) \geqslant p$
$\mu_n([x_p, \infty)) \geqslant 1 - p$.

Empirical distributions – cont (2)

5. *A* **Sample mean** *for* $X_1, X_2, \ldots, X_n$
is equal to $m = \frac{X_1+X_2+\ldots+X_n}{n}$,
i.e. the arithmetic mean of $X_1, X_2, \ldots, X_n$.

6. *A* **sample variance** *for*
$X_1, X_2, \ldots, X_n$ *is equal to*
$s^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - m)^2$,
where m *is the sample mean.*

the mean and the variance of the empirical distribution

Random vectors

1. A random vector $(X_1, X_2, ..., X_n)$
2. The joint distribution of a random vector:

The (joint) **distribution of a random vector** $X = (X_1, X_2, \ldots, X_n)$ *is a probability measure* μ_X *defined over* $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, *such that* $\mu_X(A) = \mathbb{P}(X \in A)$.

3. Marginal distributions:

$$\mu_{X_i}(B) = \mathbb{P}(X_i \in B) \text{ for } B \subseteq \mathbb{R},$$

such that for $A = \underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{i-1} \times B \times \underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{n-i}$

we have

$$\mathbb{P}(X_i \in B) = \mathbb{P}((X_1, X_2, \ldots, X_n) \in A) = \mu_X(A).$$

Random vectors – cont.

4. Example: joint distribution is more than the aggregate of marginal distributions.
5. Cumulative distribution function:

The **cumulative distribution function** *of a random vector* (X, Y) *is a function* $F_{(X,Y)} : \mathbb{R}^2 \to [0, 1]$, *such that* $F_{(X,Y)}(s, t) = \mathbb{P}(X \leqslant s, Y \leqslant t)$.

6. No simple definitions of quantiles...

Random vectors – types.

7. A discrete RV

A random vector (X,Y) *is* **discrete**, *if there exists a countable set* $S \subseteq \mathbb{R}^2$, *such that* $\mu_{(X,Y)}(S) = 1$.

8. Components are also discrete, marginals obtained by summation

9. A continuous RV

A random vector (X,Y) *is* **continuous**, *if there exists a density function, i.e. a function* $g : \mathbb{R}^2 \to [0,\infty)$, *such that for any* $A \in \mathcal{B}(\mathbb{R}^2)$, *we have* $\mu_{(X,Y)}(A) = \iint_A g(x,y)dxdy$.