# Probability Calculus

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## EXPECTED VALUE – cont.

# Plan for today

- Expected value for discrete random variables – cont.
- Expected value for continuous random variables
- Properties of the EX operator

# Expected value – discrete RV – reminder

1. Definition of expected value for discrete RV

*Let $X$ be a random variable with a discrete distribution, concentrated on $S \subset \mathbb{R}$, and let $p_x = \mathbb{P}(X = x)$ for $x \in S$. We will say that the expected value of $X$ is finite if $\sum_{x \in S} |x| p_x < \infty$. Then we can define this* **expected value** *of $X$ as $\mathbb{E}X = \sum_{x \in S} x p_x$.*

mean value, depends on the distribution only
for a finite set *S*, the E*X* always exists

# Expected value – discrete RV. cont.

## 2. Examples of calculations

- single-valued RV
- die roll
- Binomial distribution (n,p)
- variables without EX:
  - series does not converge at all
  - series does not converge absolutely

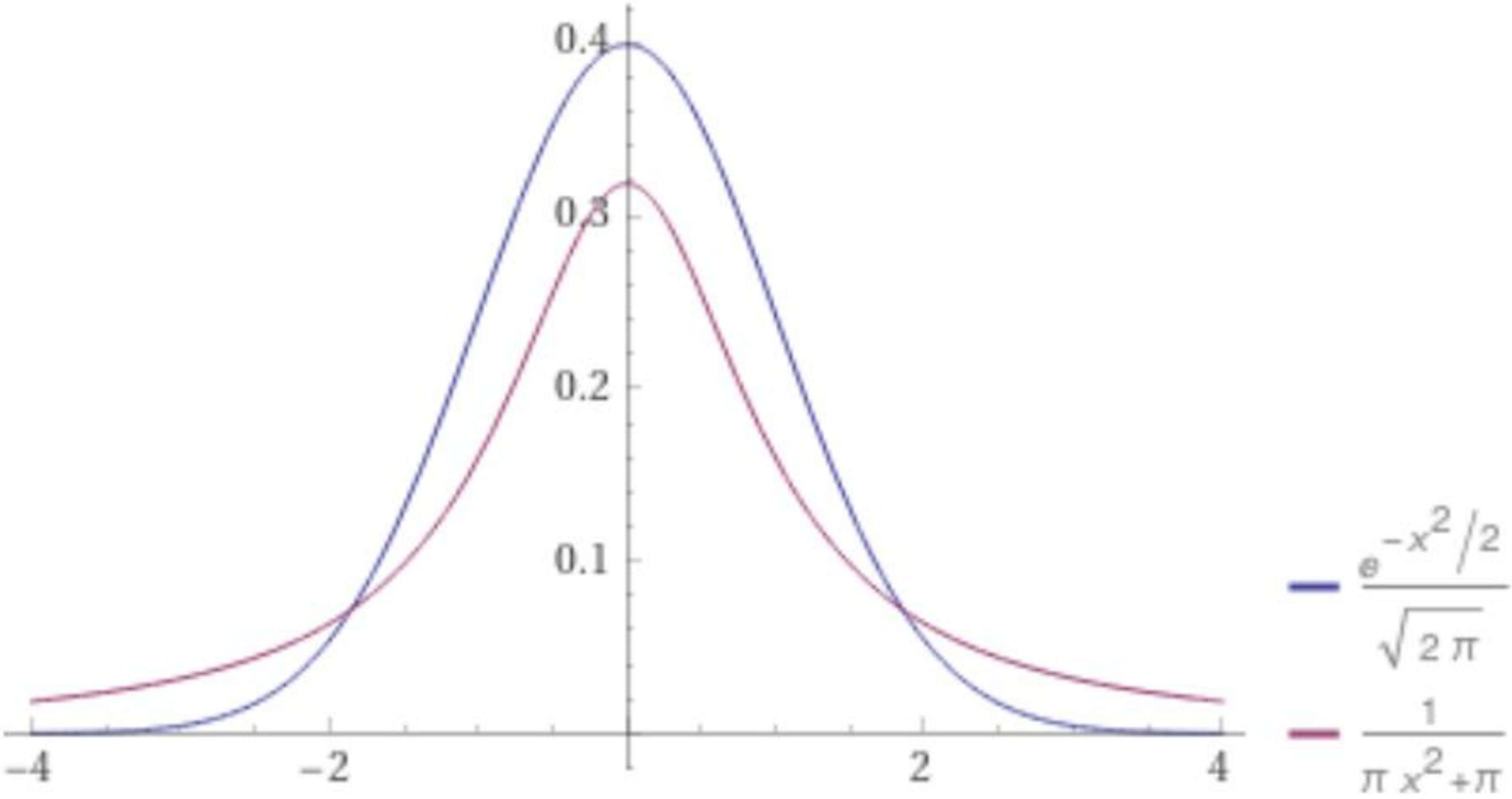
# Expected value – continuous RV

3. Definition of expected value for continuous RV

   *Let $X$ be a random variable with density $g$. If $\int_{\mathbb{R}} |x| g(x) dx < \infty$, then we will say the expected value of $X$ exists. We define this* **expected value** *of $X$ as $\mathbb{E}X = \int_{\mathbb{R}} x g(x) dx$.*

4. EX for a limited RV always exists
5. Examples of calculations
   - uniform distribution
   - standard normal distribution
   - Cauchy distribution

# Gaussian and Cauchy densities



# Expected value – properties (all RV)

## 6. Properties of E*X*

*Let* $X$ *and* $Y$ *be random variables with expected values.*

*(i) If* $X \geqslant 0$*, then* $\mathbb{E}X \geqslant 0$.

*(ii) If* $X \leqslant Y$*, then* $\mathbb{E}X \leqslant \mathbb{E}Y$.

*(iii)* $\mathbb{E}X \leqslant \mathbb{E}|X|$.

*(iv) If* $a, b \in \mathbb{R}$*, then* $aX + bY$ *has an expected value and* $\mathbb{E}(aX + bY) = a\mathbb{E}X + b\mathbb{E}Y$.

*(v) If* $X = 1_A$*, then* $\mathbb{E}X = \mathbb{P}(A)$.

## 7. Generalization of *(iv)*

## 8. Examples

# Expected value of a function of a RV

## 9. Theorem

*Let* $\phi : \mathbb{R} \to \mathbb{R}$ *be a Borel function.*
*(i) If* $X$ *is discrete, concentrated on the set* $S$*, and* $p_x = \mathbb{P}(X = x)$ *for* $x \in S$*, then the random variable* $\phi(X)$ *has an expected value if and only if* $\sum_{x \in S} |\phi(x)| p_x < \infty$*, and the expected value is equal to* $\mathbb{E}\phi(X) = \sum_{x \in S} \phi(x) p_x$.
*(ii) If* $X$ *is continuous with density* $g$*, then the random variable* $\phi(X)$ *has an expected value if and only if* $\int_R |\phi(x)| g(x) dx < \infty$*, and the expected value is equal to* $\mathbb{E}\phi(X) = \int_R \phi(x) g(x) dx$.

## 10. Examples

## Expected value of a non-negative RV

11. Calculating E*X* based on the CDF:

for non-negative integer values

$$\mathbb{E}X = \sum_{k=0}^{\infty} k\mathbb{P}(X = k) = \sum_{k=1}^{\infty} k\mathbb{P}(X = k).$$

$$\begin{aligned} = \; & \mathbb{P}(X = 1) + \\ & \mathbb{P}(X = 2) + \; \mathbb{P}(X = 2) + \\ & \mathbb{P}(X = 3) + \; \mathbb{P}(X = 3) + \; \mathbb{P}(X = 3) + \\ & \mathbb{P}(X = 4) + \; \mathbb{P}(X = 4) + \; \mathbb{P}(X = 4) + \; \mathbb{P}(X = 4) + \\ & \ldots \end{aligned}$$

and eventually:

$$\mathbb{E}X = \sum_{k=1}^{\infty} \mathbb{P}(X \geqslant k) = \sum_{k=0}^{\infty} \mathbb{P}(X > k).$$

# Expected value – cont

12. Calculating EX based on the CDF – general case of non-negative RV

*Let* $X$ *be a non-negative random variable.*
*(i) If* $\int_0^\infty \mathbb{P}(X > t)dt < \infty$*, then* $X$ *has an expected value and* $\mathbb{E}X = \int_0^\infty \mathbb{P}(X > t)dt$.
*(ii) If* $p \in (0, \infty)$ *and* $\int_0^\infty pt^{p-1}\mathbb{P}(X > t)dt < \infty$*, then* $X^p$ *has an expected value and*
$\mathbb{E}X^p = \int_0^\infty pt^{p-1}\mathbb{P}(X > t)dt$.

13. Examples

- geometric distribution
- exponential distribution
- *p*-th moments
- non-discrete non-continuous RV

## Expected value: summary

1. Mean value
2. For discrete RV: weighted average
3. For continuous RV: average weighted by density
4. Linear operator
5. Calculations for non-negative RV
6. Calculating $\mathrm{E}\phi(X)$