Probability Calculus

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CUMULATIVE DISTRIBUTION FUNCTION, EXPECTED VALUE – INTRO

Plan for today

- Continuous RV – cont.
- Cumulative Distribution Functions
- Transformations of random variables
- Quantiles
- Expected value for discrete random variables

Continuous Random variable examples – cont.

Examples of continuous random variables

- uniform distribution
- exponential distribution
- standard normal distribution
- (General) normal distribution

Random variables – the CDF

1. The definition of a CDF

The Cumulative distribution function *of a random variable* $X : \Omega \to \mathbb{R}$ *is a function* $F_X : \mathbb{R} \to [0, 1]$, *such that* $F_X(t) = \mathbb{P}(X \leqslant t)$.

depends on the distribution only!
→ CDF of distribution

Random variables – the CDF

2. Examples of CDFs

- Dirac delta
- Two-point distribution – discrete distribution
- Exponential distribution
- Normal distribution – no simple form…

CDFs

3. Properties of the CDF

The cumulative distribution function F_X *of a random variable* X *has the following properties:*

(i) F_X *is nondecreasing,*

(ii) $\lim_{t\to\infty} F_X(t) = 1$ *and* $\lim_{t\to-\infty} F_X(t) = 0$,

(iii) F_X *is right-continuous.*

4. CDF → distribution

For any function $F : \mathbb{R} \to \mathbb{R}$ *satisfying the conditions (i)-(iii) above, there exists a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and a random variable* $X : \Omega \to \mathbb{R}$ *such that* F *is the CDF of* X*. Furthermore, the distribution of* X *is determined unequivocally.*

CDFs – cont.

4. Further properties of the CDF:

If F_X *is a cumulative distribution function of a random variable* X*, then for all* $t \in \mathbb{R}$ *we have*
$F_X(t-) = \mathbb{P}(X < t)$ *and* $F_X(t) - F_X(t-) = \mathbb{P}(X = t)$.
In particular, if F_X *is continuous at point* t*, then* $\mathbb{P}(X = t) = 0$.

CDFs – cont (2)

5. CDF → density

Let F be the CDF of a random variable X.

1. If F is not continuous, then X does not have a continuous distribution (does not have a density function).

2. Assume F is continuous. If F is differentiable apart from a finite set of points, then the function

$$g(t) = \begin{cases} F'(t) & \text{if } F'(t) \text{ exists,} \\ 0 & \text{otherwise,} \end{cases}$$

is a density function for X.

6. Examples

- uniform distribution
- distribution that is neither discrete nor continuous

Transformation of random variables

1. Well-behaved transformations of continuous variables

Assume X *is a random variable with density* f.
If the values of X *fall within the interval* (a, b)
(with probability 1), and $\varphi : (a, b) \to \mathbb{R}$ *is* C^1
and $\varphi'(x) \neq 0$ *for* $x \in (a, b)$, *then*
$Y = \varphi(X)$ *is continuous with a density function*
$g(y) = f(h(y))|h'(y)|1_{\varphi((a,b))}(y)$,
where $h(s) = \varphi^{-1}(s)$.

2. Example

Quantiles

1. Definition

Let X *be a random variable and* $p \in [0,1]$.
A **quantile of rank** p *of the variable* X *is any value* x_p, *such that*
$\mathbb{P}(X \leqslant x_p) \geqslant p$ *and*
$\mathbb{P}(X \geqslant x_p) \geqslant 1-p$.

2. Examples
   - continuous distribution (N(0,1))
   - discrete distribution

Expected value – discrete RV

1. Motivation & intuition
2. Definition of expected value for discrete RV

Let X be a random variable with a discrete distribution, concentrated on $S \subset \mathbb{R}$, and let $p_x = \mathbb{P}(X = x)$ for $x \in S$. We will say that the expected value of X is finite if $\sum_{x \in S} |x| p_x < \infty$. Then we can define this **expected value** *of X as $\mathbb{E}X = \sum_{x \in S} x p_x$.*

mean value, depends on the distribution only
for a finite set *S*, the E*X* always exists

Expected value – discrete RV. cont.

3. Examples of calculations

- single-valued RV
- die roll
- Binomial distribution (n,p)
- variables without EX:
  - series does not converge at all
  - series does not converge absolutely