Probability Calculus

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RANDOM VARIABLES – INTRO:

Plan for today

- Poisson theorem
- Definition of the distribution of a random variable
- Description of the distribution of a random variable – examples
- Cumulative Distribution Function – intro.

Poisson Theorem

1. Poisson Theorem

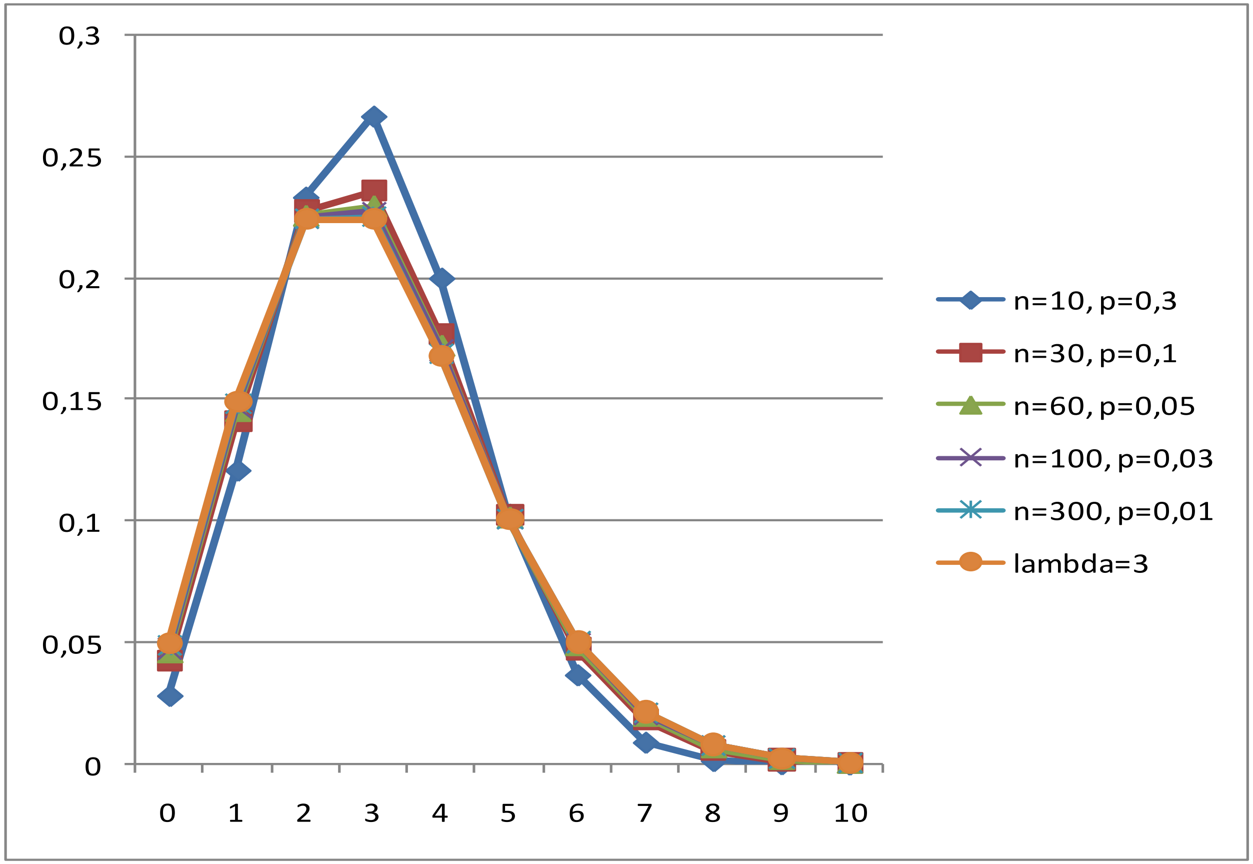
   If $p_n \in [0,1]$, $\lim_{n\to\infty} np_n = \lambda > 0$,
   then for $k = 0,\ 1,\ 2,\ \ldots,$
   we have that
   $\lim_{n\to\infty} \binom{n}{k} p_n^k (1-p_n)^{n-k} = \frac{\lambda^k}{k!} e^{-\lambda}.$

2. Assessment of approximation error

   Let S_n *denote the number of successes in a Bernoulli process with* n *trials and a probability of success in a single trial equal to* p, *and let* $\lambda = np$. *For any* $A \subset \{0, 1, 2, \ldots\}$, *we have*
   $\left|\mathbb{P}(S_n \in A) - \sum_{k\in A} \frac{\lambda^k}{k!} e^{-\lambda}\right| \leqslant \frac{\lambda^2}{n}.$

Poisson Theorem – cont.

The Poisson and Bernoulli processes



Random variables – basics

1. Motivation – functions of the results of an experiment
2. Definition of a **random variable**

   A real-valued **random variable** *is any function* $X : \Omega \to \mathbb{R}$, *such that for all* $a \in \mathbb{R}$ *the set* $X^{-1}((-\infty, a])$ *is an event, i.e.* $X^{-1}((-\infty, a]) \in \mathcal{F}$.

   $X^{-1}((-\infty, a]) = \{\omega \in \Omega : X(\omega) \leq a\}$

3. Examples
   - number of heads
   - sum of points on dice
   - the distance to a given point

Random variables – distribution

4. Functions of random variables
5. Examples of descriptions of random variables.
6. Definition of a random v. **distribution**

   The **probability distribution of a random variable** X *(real-valued) is the probability* μ_X *on* $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, *such that* $\mu_X(A) = \mathbb{P}(X \in A)$.
7. Different r.v. have the same distributions

notation: $X \sim \mu$

we forget about Ω

Random variables – examples

8. Examples of random variables

- die roll
- **discrete distributions**
- Binomial distribution
- Geometric distribution
- Poisson distribution
- uniform distribution over an interval: a **continuous distribution**
- another continuous distribution

Continuous random variables

9. Definition of a **continuous random variable** and a **density function**

A random variable X has a **continuous distribution**, *if there exists a function* $g : \mathbb{R} \to \mathbb{R}_+$, *such that for any set* $A \in \mathcal{B}(\mathbb{R})$,

$\mu_X(A) = \mathbb{P}(X \in A) = \int_A g(x)dx$.

g is called the **probability density function of** X.

10. The properties of density functions

- nonnegative
- normalized
- determines the distribution unequivocally

Random variable examples – cont.

11. More examples of continuous random variables
    - uniform distribution
    - exponential distribution
    - standard normal distribution
    - general normal distribution
    - (Dirac delta)

Random variables – the CDF

1. The definition of a CDF

The Cumulative distribution function *of a random variable* $X : \Omega \to \mathbb{R}$ *is a function* $F_X : \mathbb{R} \to [0, 1]$*, such that* $F_X(t) = \mathbb{P}(X \leqslant t)$.

depends on the distribution only!
→ CDF of distribution

Random variables – the CDF

2. Examples of CDFs

- Dirac delta
- Two-point distribution – discrete distribution
- Exponential distribution
- Normal distribution – no simple form…

CDFs

3. Properties of the CDF

The cumulative distribution function F_X *of a random variable* X *has the following properties:*
(i) F_X *is nondecreasing,*
(ii) $\lim_{t\to\infty} F_X(t) = 1$ *and* $\lim_{t\to-\infty} F_X(t) = 0$,
(iii) F_X *is right-continuous.*

4. CDF → distribution

For any function $F : \mathbb{R} \to \mathbb{R}$ *satisfying the conditions (i)-(iii) above, there exists a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and a random variable* $X : \Omega \to \mathbb{R}$ *such that* F *is the CDF of* X*. Furthermore, the distribution of* X *is determined unequivocally.*