Probability Calculus

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INDEPENDENCE OF EVENTS

BERNOULLI PROCESS

POISSON THEOREM

Plan for today

1. Bayes theorem – cont.
2. Independence of events
3. The Bernoulli Process
4. Approximation of the Bernoulli Process for large *n* – Poisson Theorem

Conditional probability – cont.

Theorem (Bayes' Rule)

Let $\{H_i\}_{i\in I}$ *be a countable (finite or infinite) partition of* Ω *into sets of positive probability. For any event* A *of positive probability, we have*

$$\mathbb{P}(H_j|A) = \frac{\mathbb{P}(A|H_j)\mathbb{P}(H_j)}{\sum_{i\in I}\mathbb{P}(A|H_i)\mathbb{P}(H_i)}.$$

Examples

Independence of Events

Intuitions

Independence of Events

1. Definition

$$\textit{Events } A \textit{ and } B \textit{ are } \textbf{independent},$$
$$\textit{if } \mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot P(B).$$

Symmetric. Stochastic independence

2. Examples
   - die roll
   - choosing a card

Independence of Events – cont.

3. Independence of 3+ events

Events $A_1, A_2, \ldots, A_n$ *are* **independent**, *if for all indices* $1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n$, $k = 2, 3, \ldots, n$, *we have*

$$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}) = \mathbb{P}(A_{i_1}) \cdot \mathbb{P}(A_{i_2}) \cdot \ldots \cdot \mathbb{P}(A_{i_k}).$$

4. Examples.

- The definition may not be simplified!
- Independence and pairwise independence

Independence of Events – cont. (2)

5. Theorem. Independence conditions

Let $A_1, A_2, \ldots, A_n$ *be a sequence of events, and denote* $A_i^0 = A_i$, $A_i^1 = A_i'$.
The following conditions are equivalent:

(i) *events* $A_1, A_2, \ldots, A_n$ *are independent,*

(ii) *for any sequence* $\varepsilon_1, \ldots, \varepsilon_n$, *where* $\varepsilon_i \in \{0, 1\}$ $(i = 1, \ldots, n)$, *events* $B_1 = A_1^{\varepsilon_1}, \ldots, B_n = A_n^{\varepsilon_n}$ *are independent,*

(iii) *for any sequence* $\varepsilon_1, \ldots, \varepsilon_n$, *where* $\varepsilon_i \in \{0, 1\}$ $(i = 1, \ldots, n)$, *we have* $\mathbb{P}(A_1^{\varepsilon_1} \cap \ldots \cap A_n^{\varepsilon_n}) = \mathbb{P}(A_1^{\varepsilon_1}) \cdot \ldots \cdot \mathbb{P}(A_n^{\varepsilon_n})$.

Bernoulli Process

1. Definition

A **Bernoulli process** *is a sequence of n independent repetitions of a single experiment (referred to as a* Bernoulli trial*) with two possible outcomes: one of these outcomes is referred to as a* success *(usually denoted as 1), and occurs with probability $p \in [0, 1]$, and the other one is a failure (usually denoted as 0), and occurs with probability $q = 1 - p$.*

- a finite or an infinite process

Bernoulli Process – cont.

2. Examples
3. Probability in a Bernoulli process:

$$\Omega = \{(a_1, a_2, \ldots, a_n) : a_i \in \{0,1\}, i = 1,\ 2\ \ldots,\ n\}$$

$$\mathcal{F} = 2^{\Omega}$$

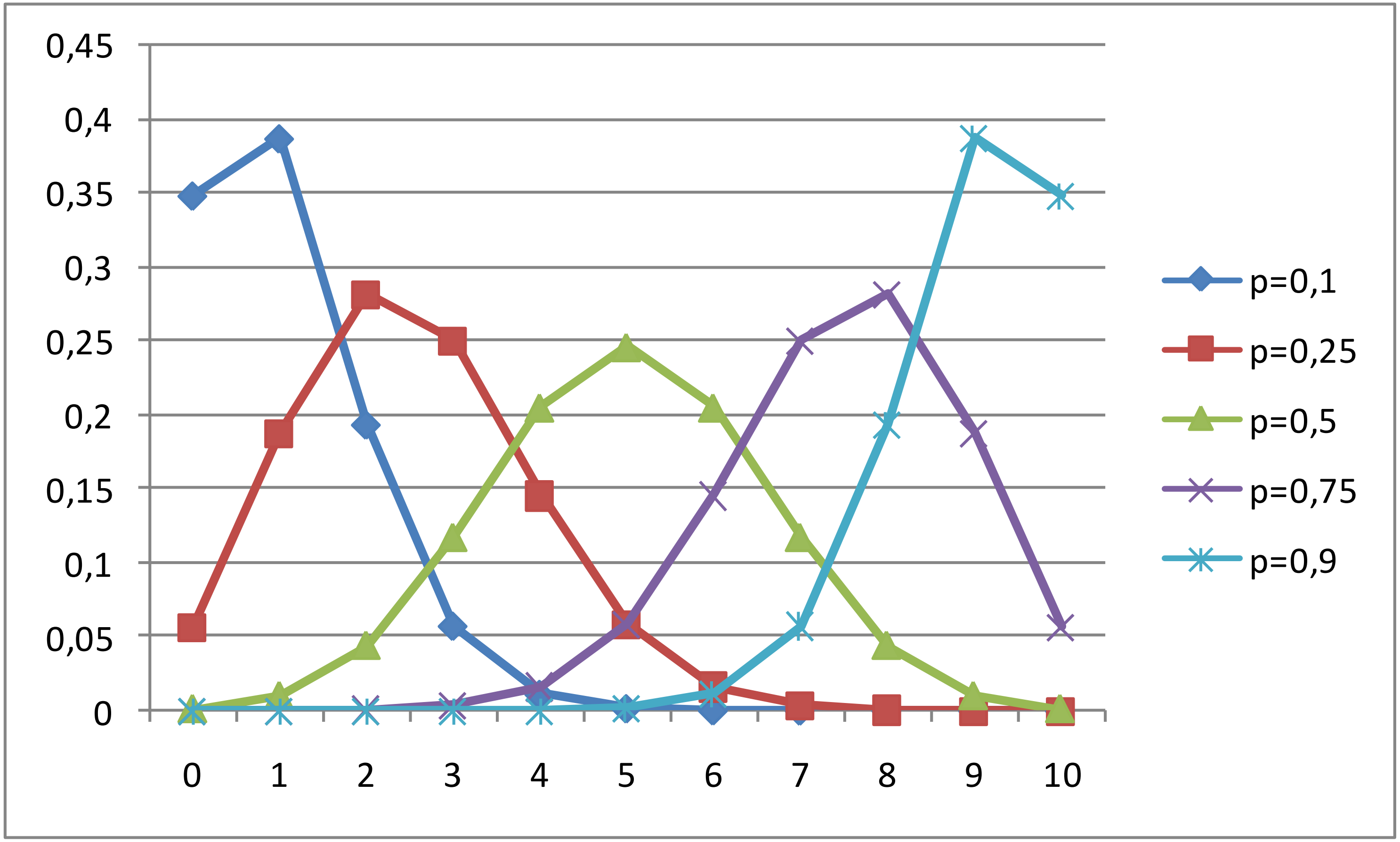
$$\mathbb{P}(\{(a_1,\ a_2,\ \ldots,\ a_n)\}) = p^{\sum_{i=1}^{n} a_i}(1-p)^{n-\sum_{i=1}^{n} a_n}$$

- probability of exactly *k* successes in *n* trials

$$\binom{n}{k} p^k (1-p)^{n-k}$$

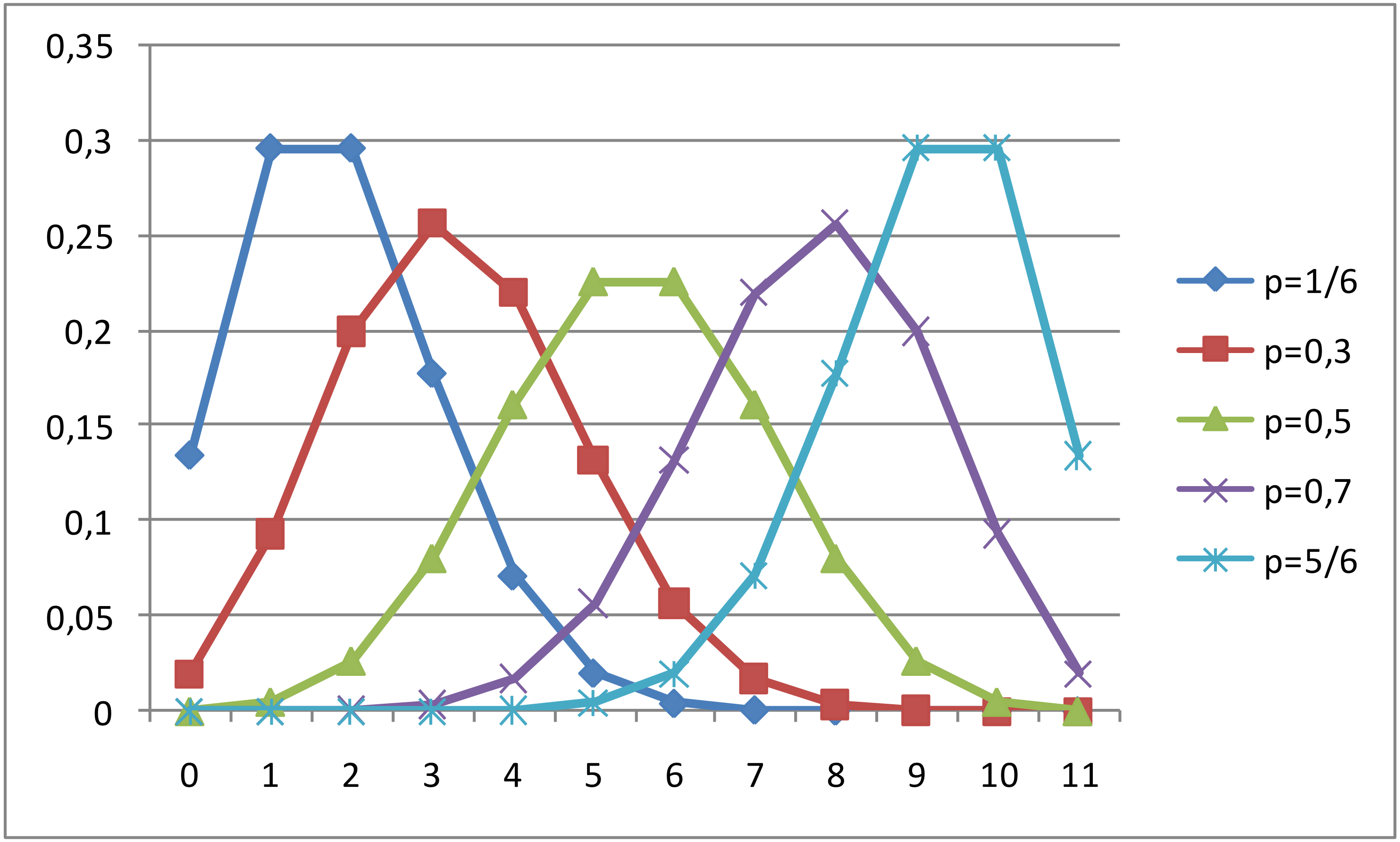
Bernoulli Process – cont. (2)

n=10



Bernoulli Process – cont. (3)

n=11



Bernoulli Process – cont. (4)

4. Examples
   - coin flip
   - die roll
5. The most probable number of successes
6. Infinite sequence of heads

Poisson Theorem

1. Poisson Theorem

   $If\ p_n \in [0,1],\ \lim_{n\to\infty} np_n = \lambda > 0,$
   $then\ for\ k = 0,\ 1,\ 2,\ \ldots,$
   $we\ have\ that$
   $\lim_{n\to\infty} \binom{n}{k} p_n^k (1-p_n)^{n-k} = \frac{\lambda^k}{k!} e^{-\lambda}.$

2. Assessment of approximation error

   Let S_n *denote the number of successes in a Bernoulli process with* n *trials and a probability of success in a single trial equal to* p, *and let* $\lambda = np$. *For any* $A \subset \{0, 1, 2, \ldots\}$, *we have*
   $\left|\mathbb{P}(S_n \in A) - \sum_{k\in A} \frac{\lambda^k}{k!} e^{-\lambda}\right| \leqslant \frac{\lambda^2}{n}.$

Poisson Theorem – cont.

The Poisson and Bernoulli processes