Probability Calculus

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INTRODUCTION TO PROBABILITY – CONT.

CONDITIONAL PROBABILITY

Plan for today

1. Sample spaces and basic properties of probability – cont.
2. Conditional probability

REMINDER:
Probability formally - Kolmogorov Axioms

- For a given (Ω, $\mathcal{F}$) we define probability as a function satisfying 3 conditions

$$
\begin{aligned}
&(i) && 0 \leq \mathbb{P}(A) \leq 1, \\
&(ii) && \mathbb{P}(\Omega) = 1, \\
&(iii) && \text{if } A_1, A_2, \ldots \in \mathcal{F} \text{ are pairwise disjoint} \\
& && \mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n).
\end{aligned}
$$

- Probability space (Ω, $\mathcal{F}$, $\mathbb{P}$)

Examples

1. Symmetric coin toss, asymmetric coin toss
2. Dice rolling
3. **Classic scheme (simple probability)**
4. Drawing numbers (Eurojackpot)
5. Geometric probability
6. Coin toss until first heads

Basic properties of probability

- Theorem 1 (arithmetics)

Theorem 1. *Let* $A, B, A_1, A_2, \ldots \in \mathcal{F}$. *Then*

(i) $\mathbb{P}(\emptyset) = 0,$

(ii) *If* $A_1, A_2, \ldots, A_n$ *are pairwise disjoint, then* $\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i)$.

(iii) $\mathbb{P}(A') = 1 - \mathbb{P}(A)$.

(iv) *If* $A \subseteq B$, *then* $\mathbb{P}(B \setminus A) = \mathbb{P}(B) - \mathbb{P}(A)$.

(v) *If* $A \subseteq B$, *then* $\mathbb{P}(A) \leqslant \mathbb{P}(B)$.

(vi) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

(vii) $\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) \leqslant \sum_{i=1}^{\infty} \mathbb{P}(A_i)$.

Basic properties of probability – cont.

- Theorem 2 (inclusion-exclusion principle)

If $A_1, A_2, \ldots, A_n \in \mathcal{F}$, *then*

$$\mathbb{P}(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \ldots + (-1)^{n+1}\mathbb{P}(A_1 \cap A_2 \cap \ldots \cap A_n)$$

Further properties of probability

- Definitions of contracting and expanding sets

Assume $A_1, A_2, \ldots$ *is a sequence of events.*
We will call this sequence **expanding** *if*
$A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots,$
and **contracting** *if*
$A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots.$

Further properties of probability – cont.

- Theorem: Rule of Continuity

Assume that $(A_n)_{n=1}^{\infty}$ *is a sequence of events.*
(i) If the series is expanding, then
$\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right).$
(ii) If the series is contracting, then
$\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}\left(\bigcap_{n=1}^{\infty} A_n\right).$

Conditional probability

1. Intuition
   - New product marketing
   - Results of dice rolls when only the sum is known
2. Definition

Let X and Y be events, such that $\mathbb{P}(Y) > 0$. By a **conditional probability** *of event X under the condition Y we will understand*

$$\mathbb{P}(X|Y) = \frac{\mathbb{P}(X \cap Y)}{\mathbb{P}(Y)}.$$

Conditional probability – cont.

3. Conditional probability is probability
4. Theorem (Chain rule)

For any sequence of events $A_1, \ldots, A_n$, *such that* $\mathbb{P}(A_1 \cap A_2 \cap \ldots \cap A_{n-1}) > 0$, *we have*

$$\mathbb{P}(A_1 \cap A_2 \cap \ldots \cap A_n) =$$

$$\mathbb{P}(A_1) \cdot \mathbb{P}(A_2|A_1) \cdot \mathbb{P}(A_3|A_1 \cap A_2) \cdots \mathbb{P}(A_n|A_1 \cap A_2 \cap \ldots \cap A_{n-1}).$$

Conditional probability – cont. (2)

5. Example (Succesive draws)
6. Definition of partition

Any family of events $\{H_i\}_{i\in I}$, *such that* $H_i \cap H_j = \emptyset$ *for* $i \neq j$ *and* $\bigcup_{i\in I} H_i = \Omega$ *is called a* **partition** *of the sample space* Ω.

A finite, countable partition

Conditional probability – cont. (3)

7. Theorem (Law of Total Probability)

For any finite partition $\{H_1, H_2, \ldots, H_n\}$ *of the sample space* Ω, *such that all* H_i *have positive probability, and for any event* A, *we have* $\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A|H_i) \cdot \mathbb{P}(H_i)$.

8. Examples
   - Phone manufacturer
   - Balls in a box

Conditionat probability – cont. (4)

9. Theorem (Bayes' Rule)

Let $\{H_i\}_{i\in I}$ be a countable (finite or infinite) partition of Ω into sets of positive probability. For any event A of positive probability, we have

$$\mathbb{P}(H_j|A) = \frac{\mathbb{P}(A|H_j)\mathbb{P}(H_j)}{\sum_{i\in I}\mathbb{P}(A|H_i)\mathbb{P}(H_i)}.$$

10. Examples