Mathematical Statistics

Anna Janicka

Lecture II,1.03.2021

DESCRIPTIVE STATISTICS, PART II

Plan for today

1. Descriptive Statistics, part II:
   - median
   - mode
   - quantiles
   - measures of variability
   - measures of asymmetry
   - the boxplot

Measures of central tendency – reminder

- *Classic:*
  - *arithmetic mean*
- Position (order, rank):
  - median
  - mode
  - quartile

Example 1 – cont.

Grade	*Number*	*Frequency*
2	74	29.84%
3	76	30.65%
3.5	48	19.35%
4	31	12.50%
4.5	9	3.63%
5	10	4.03%
Total	248	100%

Mode – examples
Quartiles – example
Variance – example

Example 3 – cont.

Interval	*Class mark*	*Number*	*Frequency*	*Cumulative number* cn_i	*Cumulative frequency* cf_i
(30,40]	35	11	0,11	11	0,11
(40,50]	45	23	0,23	34	0,34
(50,60]	55	33	0,33	67	0,67
(60,70]	65	12	0,12	79	0,79
(70,80]	75	6	0,06	85	0,85
(80,90]	85	8	0,08	93	0,93
(90,100]	95	3	0,03	96	0,96
(100,110]	105	2	0,02	98	0,98
(110,120]	115	2	0,02	100	1
Total		100	1		

Mode – example
Quartiles – example
Variance – example

Median

Median

(any) number such that at least half of the observations are less than or equal to it and at least half of the observations are greater than or equal to it

□ raw data:

$$Med = \begin{cases} X_{\frac{n+1}{2}:n} & n \text{ odd} \\ \frac{1}{2}\left(X_{\frac{n}{2}:n} + X_{\frac{n}{2}+1:n}\right) & n \text{ even} \end{cases}$$

where $X_{i:n}$ is the ***i*-th order statistic**, i.e. the i-th smallest value of the sample

Median – cont.

- □ for grouped class interval data:

$$Med \cong c_L + \frac{b}{n_M}\left(\frac{n}{2} - \sum_{i=1}^{M-1} n_i\right)$$

where:

M – number of the median's class

c_L – lower end of the median's class interval

b – length of the median's class interval

Median – examples

Example 1:

$$Med = \frac{X_{124:248} + X_{125:248}}{2} = 3$$

Example 3:

M=3, n_3=33, c_L=50, b=10

$$Med \cong 50 + \tfrac{10}{33}(50-34) \approx 54.85$$

in reality: *Med* = 55.25

Mode

Mode

the value that appears most often

- □ for grouped data:

  Mo = most frequent value

- □ for grouped class interval data:

$$Mo \cong c_L + \frac{n_{Mo} - n_{Mo-1}}{(n_{Mo} - n_{Mo-1}) + (n_{Mo} - n_{Mo+1})} \cdot b$$

where

n_{Mo} – number of elements in mode’s class,

c_L, b – analogous to the median

Mode – examples

Example 1 – cont.

Example 1:

$Mo = 3$

Example 3 – cont.

Example 3:

the mode's interval is (50,60], with 33 elements

$n_{Mo} = 33$, $c_L = 50$, $b = 10$, $n_{Mo\text{-}1} = 23$, $n_{Mo+1} = 12$

$$Mo \cong 50 + \frac{33-23}{(33-23)+(33-12)} \cdot 10 \approx 53.23$$

Which measure should we choose?

□ Arithmetic mean: for typical data series (single max, monotonous frequencies)

□ Mode: for typical data series, grouped data (the lengths of the mode’s class and neighboring classes should be equal)

□ Median: no restrictions. The most robust (in case of outlier observations, fluctuations etc.)

Quantiles, quartiles

- *p*-th quantile (quantile of rank *p*): number such that the fraction of observations less than or equal to it is at least *p*, and values greater than or equal to it at least 1-*p*

- Q_1 : first quartile = quantile of rank ¼
- Second quartile = median

  = quantile of rank ½
- Q_3: Third quartile = quantile of rank ¾

Quantiles – cont.

Empirical quantile of rank *p*:

$$Q_p = \begin{cases} \dfrac{X_{np:n} + X_{np+1:n}}{2} & np \in Z \\ X_{[np]+1:n} & np \notin Z \end{cases}$$

Quartiles – cont.

- □ Quantiles for *p* = ¼ and *p* = ¾.
- □ For grouped class interval data – analogous to the median

$$Q_k \cong c_L + \frac{b}{n_{M_k}}\left(\frac{k\cdot n}{4} - \sum_{i=1}^{M_k-1} n_i\right)$$

for *k*=1 or 3

where M_1, M_3 – number of the quartile's class

b – length of quartile class interval

c_L – lower end of the quartile class interval

Quartiles – examples

Example1 – cont.

Example 1:

$$248 \cdot \tfrac{1}{4} = 62 \qquad 248 \cdot \tfrac{3}{4} = 186$$

so

$$Q_1 = \frac{X_{62:248} + X_{63:248}}{2} = 2, \qquad Q_3 = \frac{X_{186:248} + X_{187:248}}{2} = 3.5$$

Example 3 – cont.

Example 3:

$$100 \cdot \tfrac{1}{4} = 25 \qquad 100 \cdot \tfrac{3}{4} = 75$$

$M_1 = 2, \quad \mathrm{M}_3 = 4$ so

$$Q_1 \cong 40 + \frac{10}{23}(25 - 11) \approx 46{,}09 \qquad Q_3 \cong 60 + \frac{10}{12}(75 - 67) \approx 66{,}67$$

Variability measures

- Classical measures
  - variance, standard deviation
  - average (absolute) deviation
  - coefficient of variation
- Measures based on order statistics
  - range
  - interquartile range
  - *quartile deviation*
  - *coefficients of variation (based on order stats)*
  - *median absolute deviation*

Measures based on order statistics

- Range

  the most simple measure, does not take into account anything but the extreme values

$$r = X_{n:n} - X_{1:n}$$

- Inter Quartile Range (midspread, middle fifty)

  more robust than the range

$$IQR = Q_3 - Q_1$$

length of the interval that covers the middle 50% observations

may be further used to calculate **quartile deviation *Q***= *IQR*/2, and coefficients of variation V_Q = *Q*/*Med* or V_{Q1Q3} = *IQR*/(Q_3+Q_1) (quartile variation coefficient) or the typical range: [*Med* – *Q*, *Med* + *Q*]

Range, interquartile range – examples

Example 1:

$$r = 5 - 2 = 3,$$

$$IQR = 3.5 - 2 = 1.5$$

Example 3:

$$r \cong 120 - 30 = 90$$

(in reality $118{,}9\text{-}32{,}45 = 86{,}45$)

$$IQR \cong 66{,}67 - 46{,}09 = 20{,}58$$

Classical measures of dispersion

Variance

- □ raw data

$$\hat{S}^2 = \tfrac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2 = \tfrac{1}{n}\sum_{i=1}^{n} X_i{}^2 - (\overline{X})^2$$

- □ grouped data

$$\hat{S}^2 = \tfrac{1}{n}\sum_{i=1}^{k} n_i(X_i - \overline{X})^2 = \tfrac{1}{n}\sum_{i=1}^{k} n_i X_i{}^2 - (\overline{X})^2$$

- □ grouped class interval data

$$\hat{S}^2 \cong \tfrac{1}{n}\sum_{i=1}^{k} n_i(\overline{c}_i - \overline{X})^2 = \tfrac{1}{n}\sum_{i=1}^{k} n_i \overline{c}_i{}^2 - (\overline{X})^2$$

+ Sheppard’s correction

$$\overline{S}^2 \cong \hat{S}^2 - \tfrac{c^2}{12}$$

c=length of class interval (for equal intervals)

in general

$$\overline{S}^2 \cong \hat{S}^2 - \tfrac{1}{12n}\sum_{i=1}^{k} n_i(c_i - c_{i-1})^2$$

Variance – examples

Example 1 – cont.

Example 1:

$$\hat{S}^2 \approx$$

$$\tfrac{1}{248}\left((2-3.06)^2\cdot 74+(3-3.06)^2\cdot 76+(3.5-3.06)^2\cdot 48+(4-3.06)^2\cdot 31+(4.5-3.06)^2\cdot 9+(5-3.06)^2\cdot 10\right)$$

$$\approx 0.71$$

Example 3:

Example 3 – cont.

$$\hat{S}^2 \approx {}^{1}\!/_{100}\cdot \quad ((35-58.7)^2\cdot 11+(45-58.7)^2\cdot 23+(55-58.7)^2\cdot 33+(65-58.7)^2\cdot 12$$

$$+(75-58.7)^2\cdot 6+(85-58.7)^2\cdot 8+(95-58.7)^2\cdot 3+(105-58.7)^2\cdot 2+(115-58.7)^2\cdot 2)$$

$$=331.31$$

$$\bar{S}^2 = 331.31 - {}^{10^2}\!/_{12} \approx 322.98$$

in reality $\hat{S}^2 = 333.85$

distrubution not normal or sample too small for Sheppard’s correction – larger errors from small sample size than from class grouping.

Standard deviation

In the same units as the initial variable

$$\hat{S} = \sqrt{\hat{S}^2}, \qquad \overline{S} = \sqrt{\overline{S}^2}$$

Example 1:

$$\hat{S} \approx 0.84\ [grade]$$

Example 3:

$$\hat{S} \approx 18.2\ [m^2]$$

Average (absolute) deviation, mean deviation

Nowadays seldom used. Simple calculations.

for raw data

$$d = \frac{1}{n}\sum_{i=1}^{n} | X_i - \overline{X} |$$

etc...

We have: $d<S$

Coefficient of variation (classical)

For comparisons of the same varaible accross populations or different variables for the same population

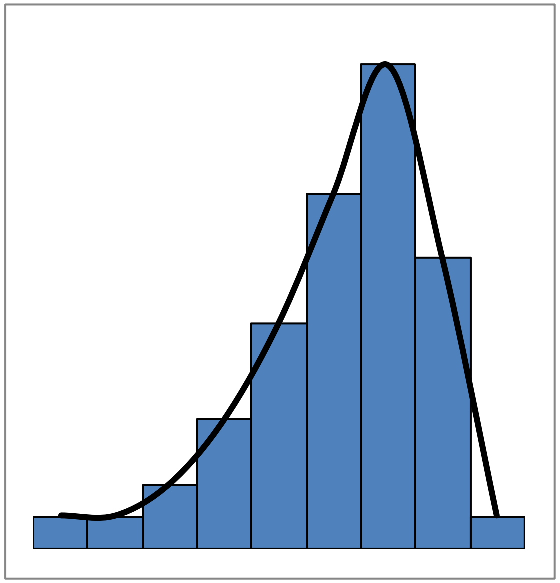
$$V_S = \frac{\hat{S}}{\overline{X}}(\cdot 100\%),$$

$$\text{or } V_d = \frac{d}{\overline{X}}(\cdot 100\%)$$

Skewness (asymmetry)

left

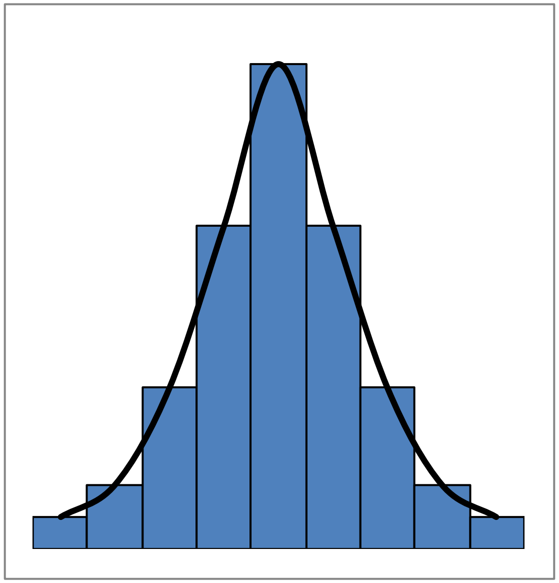
(negative)



$\bar{X} < Med < Mo$

symmetry

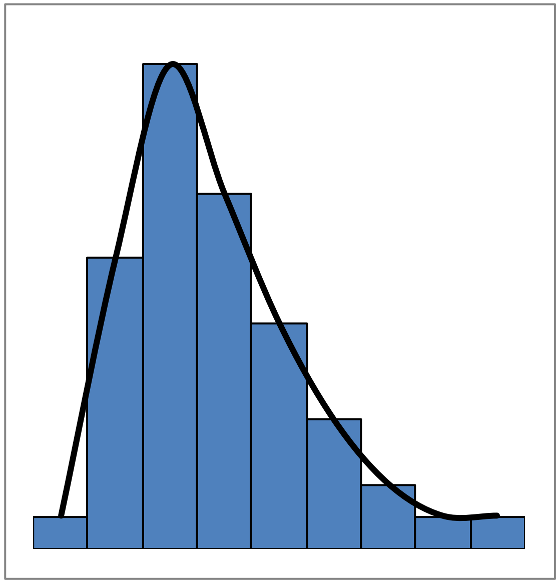
(zero)



$\bar{X} = Med = Mo$

right

(positive)



$\bar{X} > Med > Mo$

(typical order)

Measures of asymmetry

□ Skewness

$$A = \frac{M_3}{\hat{S}^3}$$

where M_3 is the third central moment

□ Skewness coefficient

$$A_1 = \frac{\overline{X} - Mo}{\hat{S}} \quad \text{or} \quad A_1 = \frac{\overline{X} - Med}{\hat{S}}$$

□ Quartile skewness coefficient

$$A_2 = \frac{Q_3 - 2Med + Q_1}{Q_3 - Q_1}$$

measures skewness for the middle 50% observations only

Interpretation

- positive values= positive asymmetry (right skewed distribution)
- negative values = negative asymmetry (left skewed distribution)
- For the skewness coefficient (with the median) and the quartile skewness coefficient the strength of asymmetry (absolute value):
  - 0 – 0.33: weak
  - 0.34 – 0.66: medium
  - 0.67 – 1: strong

Asymmetry – examples

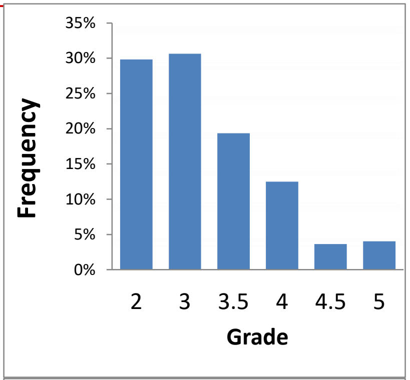
Example 1:

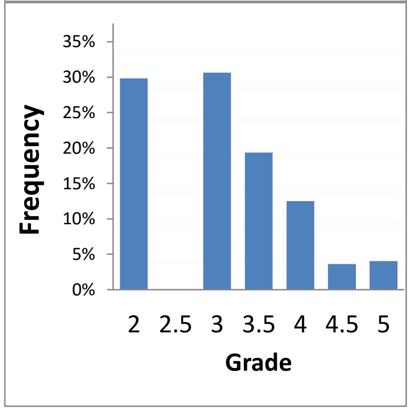
$$A \approx 0.28$$

$$A_1 = \frac{3.06-3}{0.84} \approx 0.07\,(Med)$$

$$A_1 = \frac{3.06-3}{0.84} \approx 0.07\,(Mo)$$

$$A_2 = \frac{3.5-2\cdot 3+2}{3.5-2} = -\frac{1}{3}$$



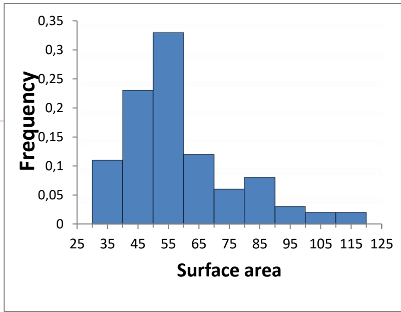


Example 3:

$$A \cong 1.15,$$

$$A_1 \cong \frac{58.7-53.23}{18.2} \approx 0.3\,(Mo)\ or\ A_1 = \frac{58.7-54.85}{18.2} \approx 0.24\,(Med)$$

$$A_2 \cong \frac{66.67-2\cdot 54.85+46.09}{66.67-46.09} \approx 0.15$$

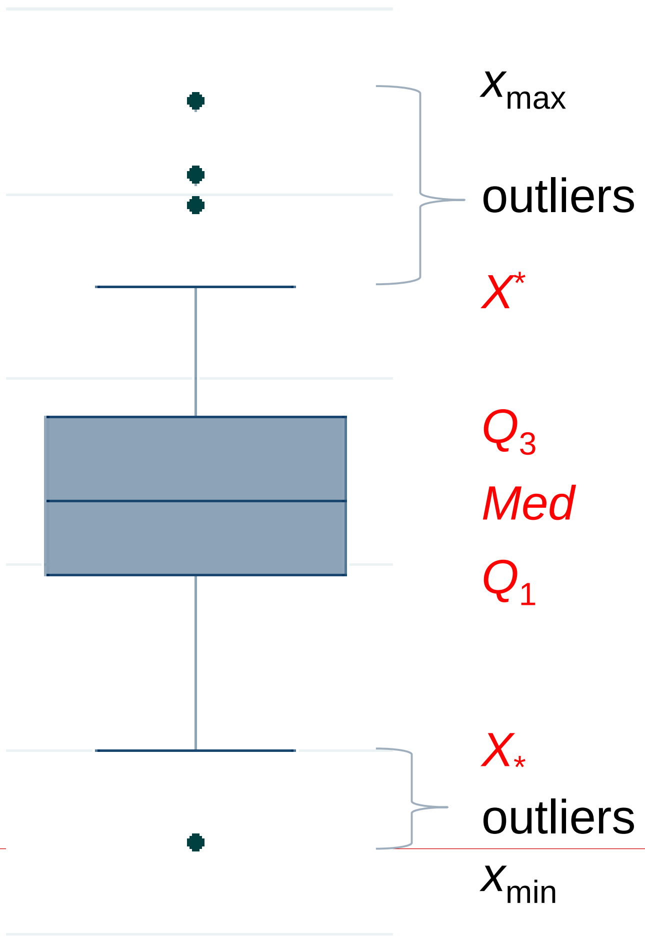


Boxplot
(Box and whisker plot)

Allows to compare two (or more) populations

$$X_* = \min\{X_i : X_i \in [Q_1 - \tfrac{3}{2} IQR, Q_1]\}$$

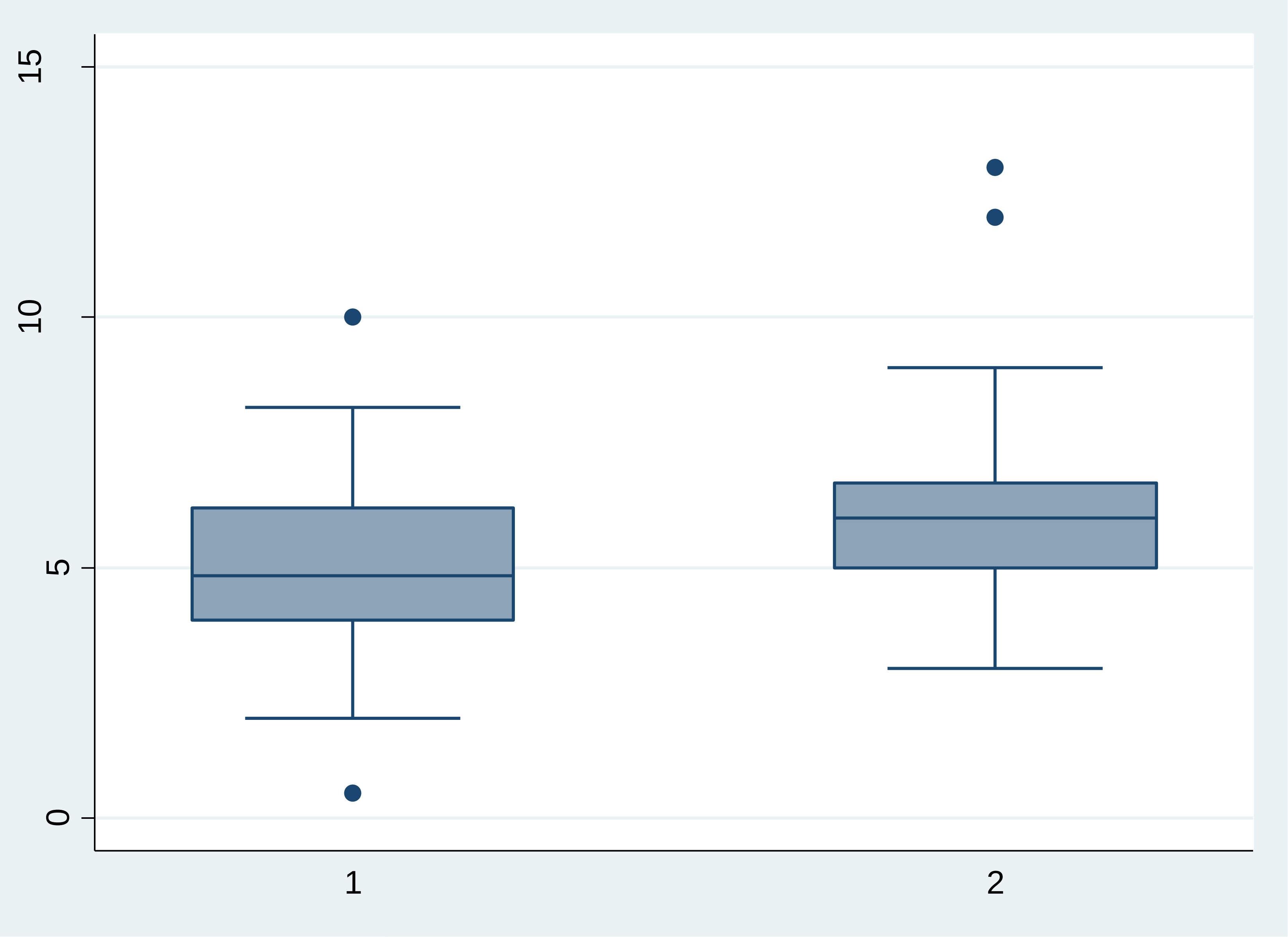
$$X^* = \max\{X_i : X_i \in [Q_3, Q_3 + \tfrac{3}{2} IQR]\}$$



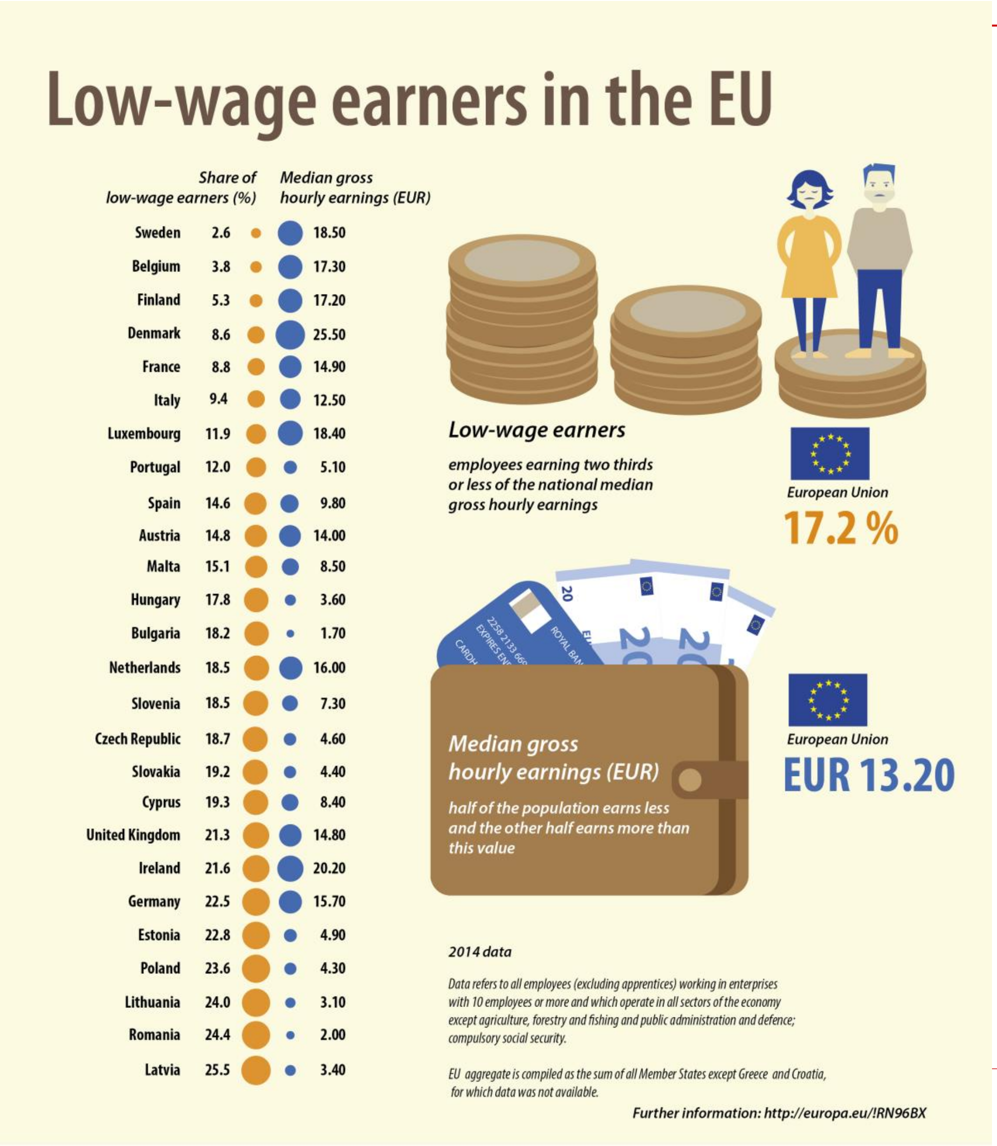
outliers:

$$x < X_* \text{ or } x > X^*$$

Boxpolot – example of comparison



Examples (1)



Low-wage earners in the EU

	Share of low-wage earners (%)	Median gross hourly earnings (EUR)
Sweden	2.6	18.50
Belgium	3.8	17.30
Finland	5.3	17.20
Denmark	8.6	25.50
France	8.8	14.90
Italy	9.4	12.50
Luxembourg	11.9	18.40
Portugal	12.0	5.10
Spain	14.6	9.80
Austria	14.8	14.00
Malta	15.1	8.50
Hungary	17.8	3.60
Bulgaria	18.2	1.70
Netherlands	18.5	16.00
Slovenia	18.5	7.30
Czech Republic	18.7	4.60
Slovakia	19.2	4.40
Cyprus	19.3	8.40
United Kingdom	21.3	14.80
Ireland	21.6	20.20
Germany	22.5	15.70
Estonia	22.8	4.90
Poland	23.6	4.30
Lithuania	24.0	3.10
Romania	24.4	2.00
Latvia	25.5	3.40

Low-wage earners

employees earning two thirds or less of the national median gross hourly earnings

European Union

17.2 %

Median gross hourly earnings (EUR)

half of the population earns less and the other half earns more than this value

European Union

EUR 13.20

2014 data

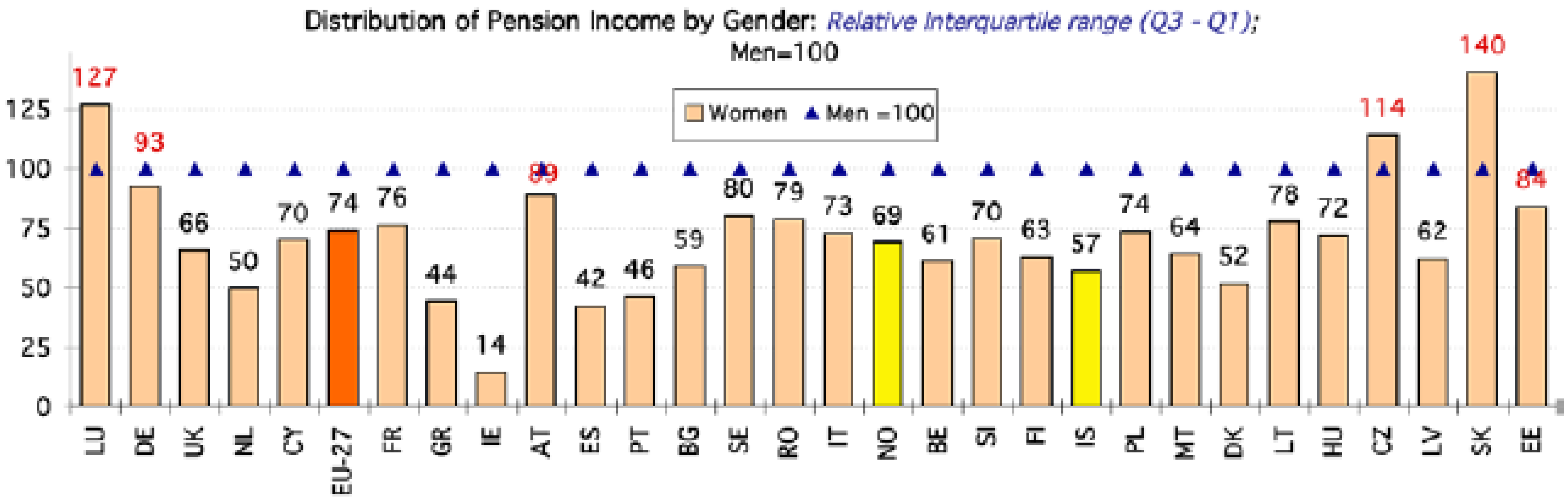
Data refers to all employees (excluding apprentices) working in enterprises with 10 employees or more and which operate in all sectors of the economy except agriculture, forestry and fishing and public administration and defence; compulsory social security.

EU aggregate is compiled as the sum of all Member States except Greece and Croatia, for which data was not available.

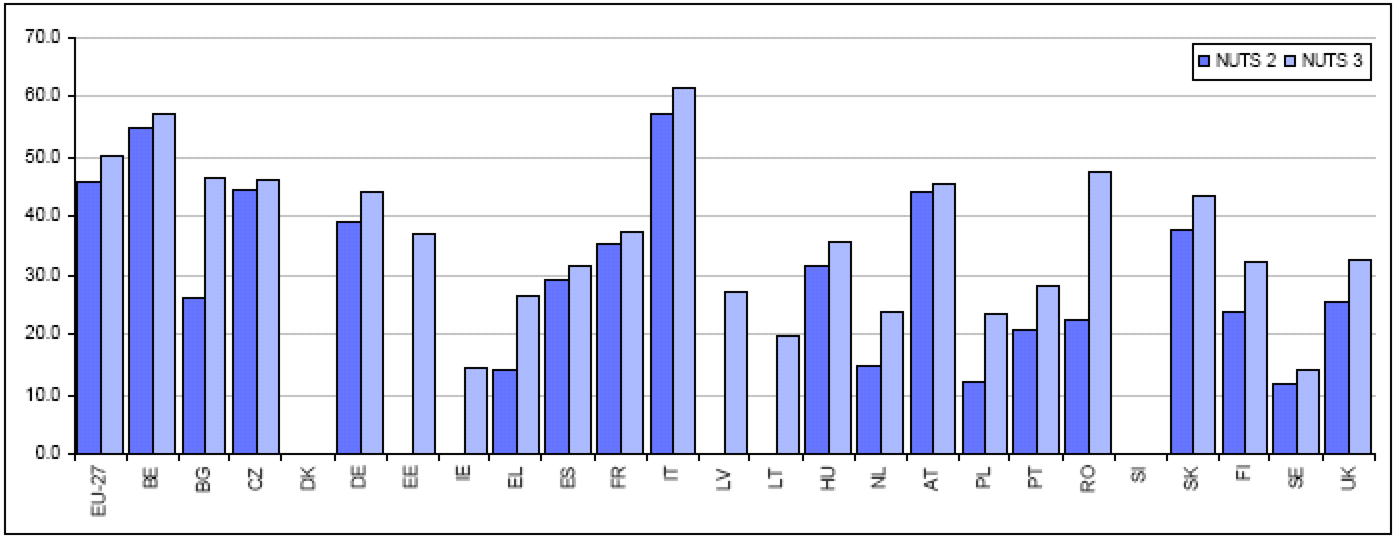
Further information: http://europa.eu/!RN96BX

ec.europa.eu/**eurostat**

Examples (2)

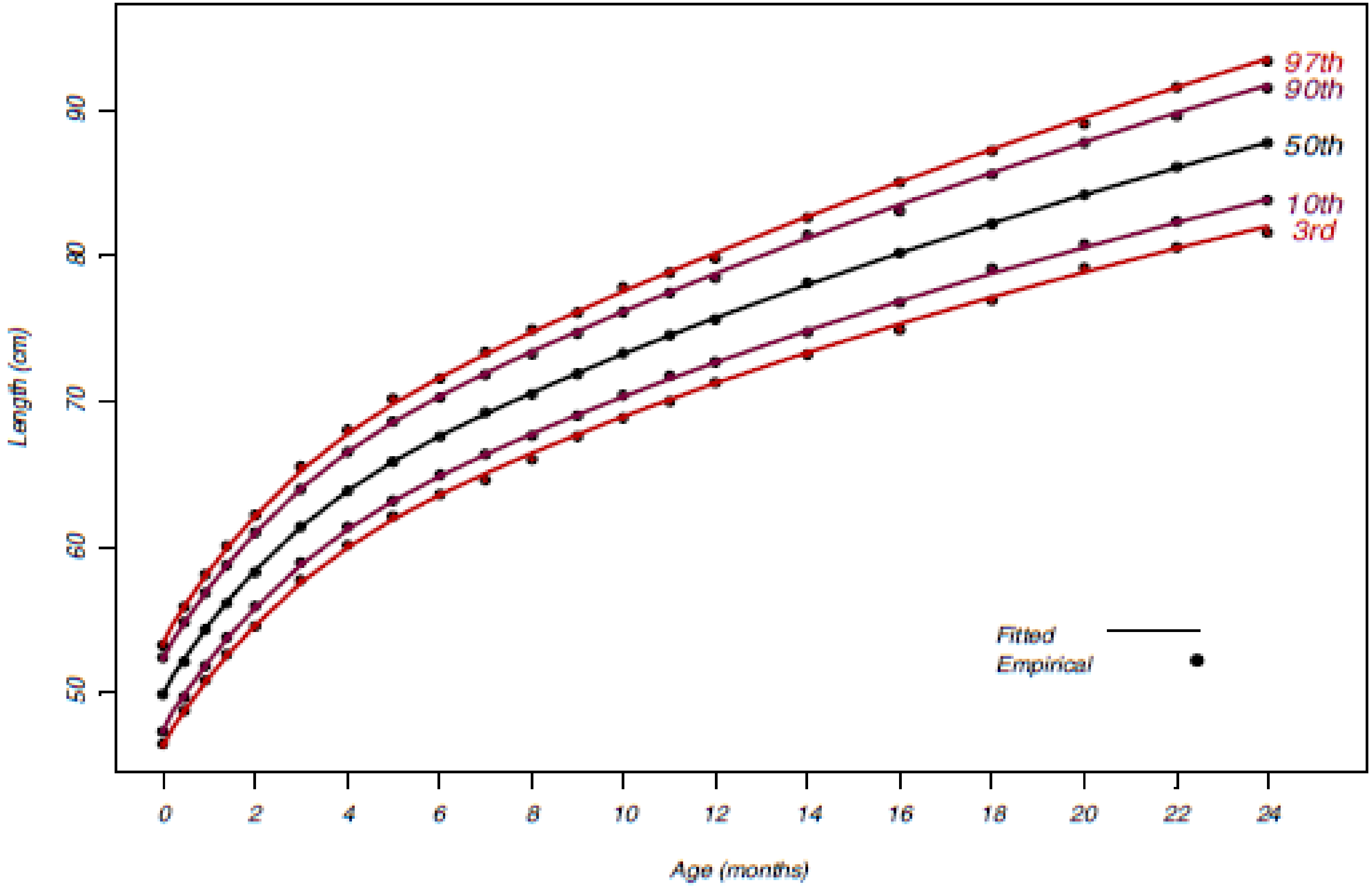


Dispersion (coeffcient of variability) of unemployment rates



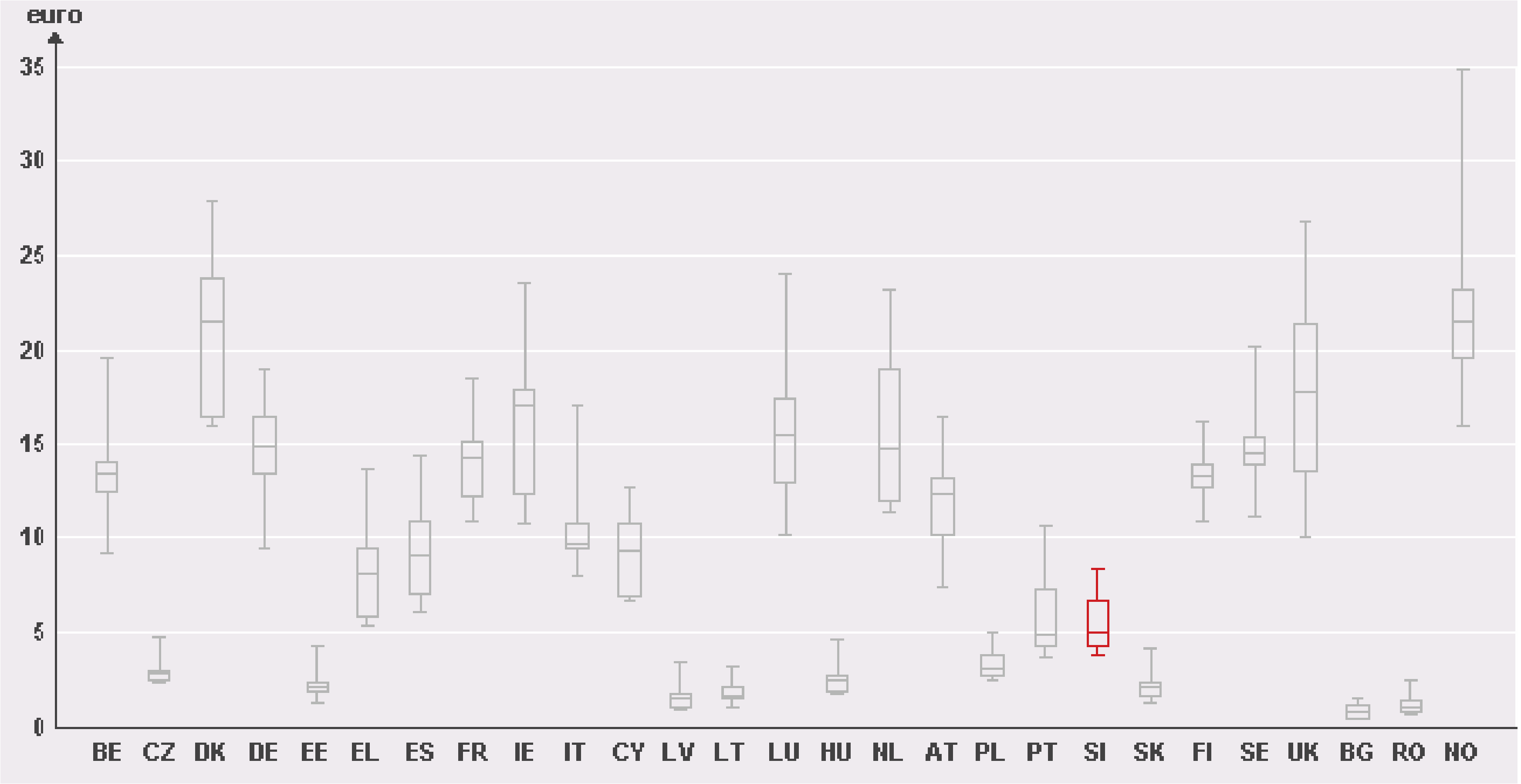
Source: European Commission

Examples (3)
Growth charts



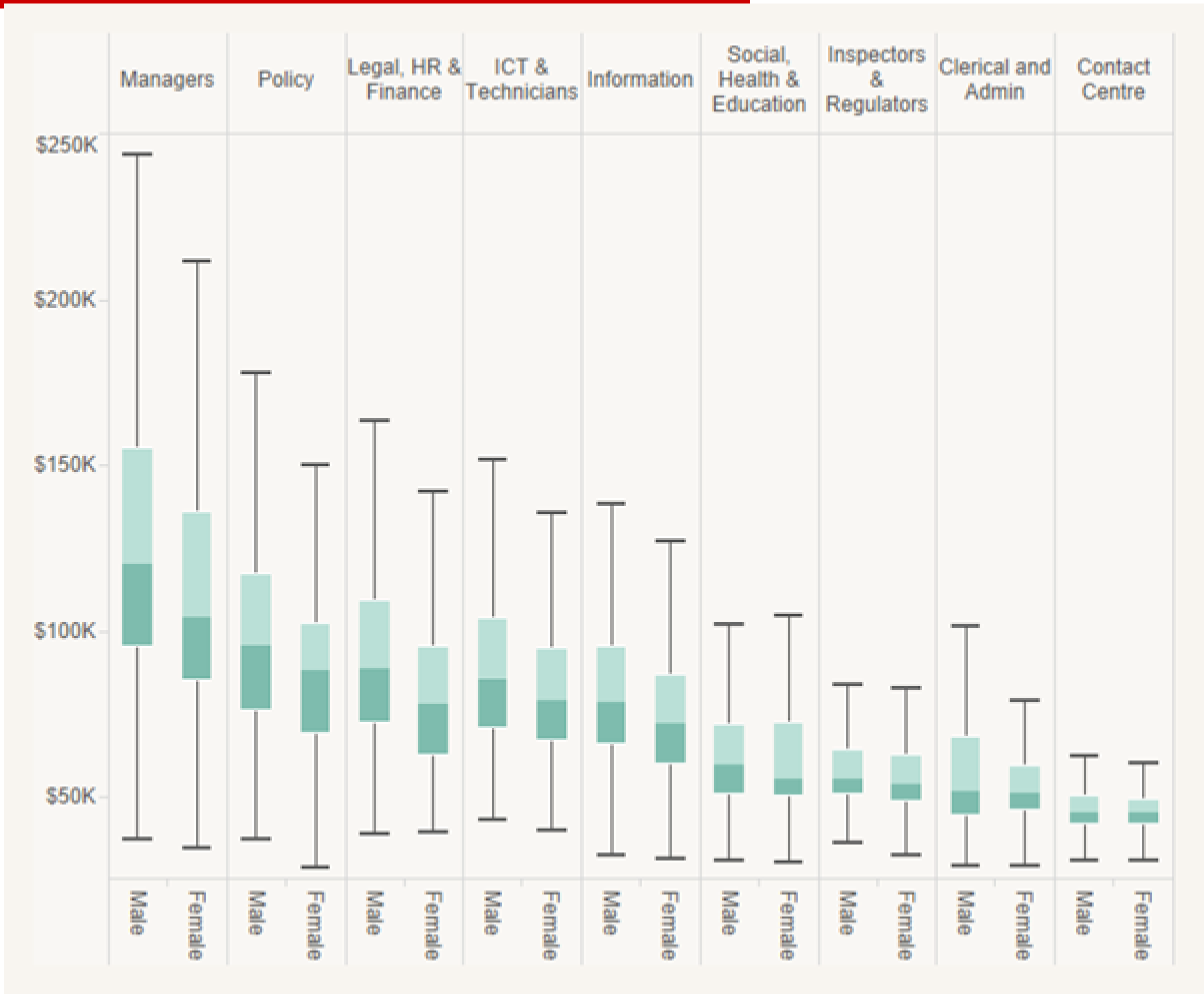
Source: WHO

Examples (4)
Gross hourly earinings



Source: European Commission

Examples(5)
Salary by occupational group and gender



Source: New Zealand State Services