**Probability Calculus Midterm Test. December 7th, 2018. Version A**

*In order to obtain the maximum number of points, you need to solve 5 out of the 7 problems below. Each problem must be solved on a separate piece of paper. Please sign each paper with your name and student's number and note the version (A, B, C, D). Each problem will be graded on a scale from 0 to 10 points. Duration: 120 minutes.*

**1.** 6 women and 6 men were randomly divided into two groups, with 8 individuals in one group and 4 individuals in the other group.

a) Calculate the probability that there will be at least two women in each of the groups.

b) Calculate the probability that there are at least five men in the 8-person group, if we know that there is at least one woman in the 4-person group.

c) Calculate the expected value of the number of women in the 8-person group.

**2.** 2% of newly manufactured processors have damaged cores. A laptop with a damaged core overheats; overheating also appears in 0.002% of laptops with fully functional cores. We assume that malfunctions in different laptops are independent.

a) A laptop overheats. What is the chance that it has a damaged core?

b) A client bought two laptops, and both of them overheated. What is the probability that at least one of them has a damaged core?

c) Using the Poisson theorem, approximate the probability that among 200 overheating laptops which were serviced in a repair shop, at most 2 were equipped with a fully functional processor.

**3.** When going to lunch, a student chooses randomly one of three cafeterias $C_1$, $C_2$, $C_3$, with probabilities $2p$, $p$ and $1-3p$, respectively ($p \in (0,1/3)$ is a fixed parameter). We assume that students choose cafeterias independently of each other and that choices on different days are also independent.

a) In order to approximate $p$, 10 students were asked how many times they chose cafeteria $C_1$ during the previous 5 days. The answers were: 0, 0, 0, 1, 1, 1, 1, 2, 2, 4. For which $p$ will the empirical mean connected with the sample be equal to the expected value of the number of choices of cafeteria $C_1$?

b) Let us assume that the queueing times in cafeterias $C_1$, $C_2$ and $C_3$ (measured in minutes) are random variables from exponential distributions with parameters 1, 1/2 and 1/3, respectively. During the next 30 days, a student plans to visit cafeteria $C_1$ 15 times, cafeteria $C_2$ 10 times, and cafeteria $C_3$ 5 times. Calculate the expected value of the total amount of time (in minutes) which will be spent on queueing during those 30 days.

**4.** Mr. Smith wants to invest a part of his savings. He can open a bank deposit (in which case his profit will amount to 2 percent per year), or he can buy shares of a company, in which case his yearly profit (in percent) will be a random variable with a CDF equal to

$$F(t) = \begin{cases} 0 & \text{if } t < 1, \\ 1 - t^{-3} & \text{if } t \geq 1. \end{cases}$$

Mr. Smith has the following strategy. He tosses a coin, for which the probability of heads amounts to 1/3; if heads appear, Mr. Smith deposits his money in the bank; otherwise, he buys shares. Let $Y$ denote Mr. Smith's profit (in percent) after a year for this strategy.

a) Find the CDF of variable $Y$. Is $Y$ continuous? Is $Y$ discrete?

b) Calculate $\mathbb{E}Y$.

**5.** Let $X$ be a random variable from a distribution with density $g(x) = 4a^{-4}x^3\mathbb{1}_{(0,a]}(x)$, where $a > 0$ is a constant.

a) Find $a$, if we know that the quantile of rank $\frac{1}{16}$ for random variable $X$ is equal to 1.

b) Calculate the expected value of variable $\frac{1}{X}$ and the variance of variable $\frac{3}{X} + 10$.

**6.** Let $X$ be a random variable from a distribution such that $\mathbb{P}(X = 1) = \mathbb{P}(X = 2) = \frac{1}{2}\mathbb{P}(X = 3) = c$.

a) Find $c$.

b) Calculate $\mathbb{P}(X \geq 2 | X \leq 2)$.

c) Let $Y$ be a variable from a normal distribution with mean 2 and variance $\lambda$. Find $\lambda$, knowing that $\mathbb{E}Y^2 = \mathbb{E}X^3$.

**7.** A businessman had two conversations between 11:00 AM and 2:00 PM. The first call started at 11:00 and lasted until hour $X$, where $X$ is a random variable from a uniform distribution over the range $[11, 14]$; the other call started at hour $X$ and lasted until 14:00.

a) What is the probability that each call lasted more than an hour?

b) Calculate the mean duration of the shorter call (in hours).

**Probability Calculus Midterm Test. December 7th, 2018. Version B**

*In order to obtain the maximum number of points, you need to solve 5 out of the 7 problems below. Each problem must be solved on a separate piece of paper. Please sign each paper with your name and student's number and note the version (A, B, C, D). Each problem will be graded on a scale from 0 to 10 points. Duration: 120 minutes.*

**1.** 7 women and 6 men were randomly divided into two groups, with 8 individuals in one group and 5 individuals in the other group.

a) Calculate the probability that there will be at least two women in each of the groups.

b) Calculate the probability that there are at least five men in the 8-person group, if we know that there is at least one woman in the 5-person group.

c) Calculate the expected value of the number of women in the 8-person group.

**2.** 2% of newly manufactured processors have damaged cores. A laptop with a damaged core overheats; overheating also appears in 0.004% of laptops with fully functional cores. We assume that malfunctions in different laptops are independent.

a) A laptop overheats. What is the chance that it has a damaged core?

b) A client bought two laptops, and both of them overheated. What is the probability that at least one of them has a fully functional core?

c) Using the Poisson theorem, approximate the probability that among 200 overheating laptops which were serviced in a repair shop, at most 3 were equipped with a fully functional processor.

**3.** When going to lunch, a student chooses randomly one of three cafeterias $C_1$, $C_2$, $C_3$, with probabilities $p$, $1-3p$ and $2p$, respectively ($p \in (0, 1/3)$ is a fixed parameter). We assume that students choose cafeterias independently of each other and that choices on different days are also independent.

a) In order to approximate $p$, 10 students were asked how many times they chose cafeteria $C_1$ during the previous 5 days. The answers were: 0, 0, 1, 1, 1, 1, 2, 2, 2, 4. For which $p$ will the empirical mean connected with the sample be equal to the expected value of the number of choices of cafeteria $C_1$?

b) Let us assume that the queueing times in cafeterias $C_1$, $C_2$ and $C_3$ (measured in minutes) are random variables from exponential distributions with parameters 1/2, 1/3 and 1, respectively. During the next 30 days, a student plans to visit cafeteria $C_1$ 10 times, cafeteria $C_2$ 5 times, and cafeteria $C_3$ 15 times. Calculate the expected value of the total amount of time (in minutes) which will be spent on queueing during those 30 days.

**4.** Mr. Smith wants to invest a part of his savings. He can open a bank deposit (in which case his profit will amount to 3 percent per year), or he can buy shares of a company, in which case his yearly profit (in percent) will be a random variable with a CDF equal to

$$F(t) = \begin{cases} 0 & \text{if } t < 2, \\ 1 - 4t^{-2} & \text{if } t \geq 2. \end{cases}$$

Mr. Smith has the following strategy. He tosses a coin, for which the probability of heads amounts to 1/4; if heads appear, Mr. Smith deposits his money in the bank; otherwise, he buys shares. Let $Y$ denote Mr. Smith's profit (in percent) after a year for this strategy.

a) Find the CDF of variable $Y$. Is $Y$ continuous? Is $Y$ discrete?

b) Calculate $\mathbb{E}Y$.

**5.** Let $X$ be a random variable from a distribution with density $g(x) = \frac{3}{8}a^{-3}x^2\mathbb{1}_{(0,2a]}(x)$, where $a > 0$ is a constant.

a) Find $a$, if we know that the quantile of rank $\frac{1}{27}$ for random variable $X$ is equal to 1.

b) Calculate the expected value of variable $\frac{1}{X}$ and the variance of variable $\frac{5}{X} - 11$.

**6.** Let $X$ be a random variable from a distribution such that $\mathbb{P}(X = 1) = 2\mathbb{P}(X = 2) = \mathbb{P}(X = 3) = c$.

a) Find $c$.

b) Calculate $\mathbb{P}(X \geq 2 | X \leq 2)$.

c) Let $Y$ be a variable from a normal distribution with mean 1 and variance $\lambda$. Find $\lambda$, knowing that $\mathbb{E}Y^2 = \mathbb{E}X^2$.

**7.** A businessman had two conversations between 10:00 AM and 1:00 PM. The first call started at 10:00 and lasted until hour $X$, where $X$ is a random variable from a uniform distribution over the range $[10, 13]$; the other call started at hour $X$ and lasted until 13:00.

a) What is the probability that each call lasted more than an hour?

b) Calculate the mean duration of the longer call (in hours).

**Probability Calculus Midterm Test. December 7th, 2018. Version C**

*In order to obtain the maximum number of points, you need to solve 5 out of the 7 problems below. Each problem must be solved on a separate piece of paper. Please sign each paper with your name and student's number and note the version (A, B, C, D). Each problem will be graded on a scale from 0 to 10 points. Duration: 120 minutes.*

**1.** 6 women and 7 men were randomly divided into two groups, with 9 individuals in one group and 4 individuals in the other group.

a) Calculate the probability that there will be at least two women in each of the groups.

b) Calculate the probability that there are at least five men in the 9-person group, if we know that there is at least one woman in the 4-person group.

c) Calculate the expected value of the number of women in the 9-person group.

**2.** 1% of newly manufactured processors have damaged cores. A laptop with a damaged core overheats; overheating also appears in 0.003% of laptops with fully functional cores. We assume that malfunctions in different laptops are independent.

a) A laptop overheats. What is the chance that it has a damaged core?

b) A client bought two laptops, and both of them overheated. What is the probability that at least one of them has a fully functional core?

c) Using the Poisson theorem, approximate the probability that among 100 overheating laptops which were serviced in a repair shop, at most 2 were equipped with a fully functional processor.

**3.** When going to lunch, a student chooses randomly one of three cafeterias $C_1$, $C_2$, $C_3$, with probabilities $1-3p$, $2p$ and $p$, respectively ($p \in (0, 1/3)$ is a fixed parameter). We assume that students choose cafeterias independently of each other and that choices on different days are also independent.

a) In order to approximate $p$, 10 students were asked how many times they chose cafeteria $C_1$ during the previous 5 days. The answers were: 0, 0, 0, 1, 1, 1, 2, 2, 3, 4. For which $p$ will the empirical mean connected with the sample be equal to the expected value of the number of choices of cafeteria $C_1$?

b) Let us assume that the queueing times in cafeterias $C_1$, $C_2$ and $C_3$ (measured in minutes) are random variables from exponential distributions with parameters 1, 1/3 and 1/2, respectively. During the next 30 days, a student plans to visit cafeteria $C_1$ 5 times, cafeteria $C_2$ 10 times, and cafeteria $C_3$ 15 times. Calculate the expected value of the total amount of time (in minutes) which will be spent on queueing during those 30 days.

**4.** Mr. Smith wants to invest a part of his savings. He can open a bank deposit (in which case his profit will amount to 4 percent per year), or he can buy shares of a company, in which case his yearly profit (in percent) will be a random variable with a CDF equal to

$$F(t) = \begin{cases} 0 & \text{if } t < 2, \\ 1 - 8t^{-3} & \text{if } t \geq 2. \end{cases}$$

Mr. Smith has the following strategy. He tosses a coin, for which the probability of heads amounts to 2/3; if heads appear, Mr. Smith deposits his money in the bank; otherwise, he buys shares. Let $Y$ denote Mr. Smith's profit (in percent) after a year for this strategy.

a) Find the CDF of variable $Y$. Is $Y$ continuous? Is $Y$ discrete?

b) Calculate $\mathbb{E}Y$.

**5.** Let $X$ be a random variable from a distribution with density $g(x) = \frac{1}{9}a^{-3}x^2\mathbb{1}_{(0,3a]}(x)$, where $a > 0$ is a constant.

a) Find $a$, if we know that the quantile of rank $\frac{1}{8}$ for random variable $X$ is equal to 1.

b) Calculate the expected value of variable $\frac{1}{X}$ and the variance of variable $\frac{2}{X} + 4$.

**6.** Let $X$ be a random variable from a distribution such that $\frac{1}{3}\mathbb{P}(X = 1) = \mathbb{P}(X = 2) = \mathbb{P}(X = 3) = c$.

a) Find $c$.

b) Calculate $\mathbb{P}(X \leq 2 | X \geq 2)$.

c) Let $Y$ be a variable from a normal distribution with mean 3 and variance $\lambda$. Find $\lambda$, knowing that $\mathbb{E}Y^2 = \mathbb{E}X^4$.

**7.** A businessman had two conversations between 12:00 PM and 3:00 PM. The first call started at 12:00 and lasted until hour $X$, where $X$ is a random variable from a uniform distribution over the range $[12, 15]$; the other call started at hour $X$ and lasted until 15:00.

a) What is the probability that each call lasted more than an hour?

b) Calculate the mean duration of the longer call (in hours).

**Probability Calculus Midterm Test. December 7th, 2018. Version D**

*In order to obtain the maximum number of points, you need to solve 5 out of the 7 problems below. Each problem must be solved on a separate piece of paper. Please sign each paper with your name and student's number and note the version (A, B, C, D). Each problem will be graded on a scale from 0 to 10 points. Duration: 120 minutes.*

**1.** 7 women and 6 men were randomly divided into two groups, with 9 individuals in one group and 4 individuals in the other group.

a) Calculate the probability that there will be at least two women in each of the groups.

b) Calculate the probability that there are at least five men in the 9-person group, if we know that there is at least one woman in the 4-person group.

c) Calculate the expected value of the number of women in the 9-person group.

**2.** 1% of newly manufactured processors have damaged cores. A laptop with a damaged core overheats; overheating also appears in 0.001% of laptops with fully functional cores. We assume that malfunctions in different laptops are independent.

a) A laptop overheats. What is the chance that it has a damaged core?

b) A client bought two laptops, and both of them overheated. What is the probability that at least one of them has a damaged core?

c) Using the Poisson theorem, approximate the probability that among 100 overheating laptops which were serviced in a repair shop, at most 3 were equipped with a fully functional processor.

**3.** When going to lunch, a student chooses randomly one of three cafeterias $C_1$, $C_2$, $C_3$, with probabilities $p$, $2p$ and $1-3p$, respectively ($p \in (0, 1/3)$ is a fixed parameter). We assume that students choose cafeterias independently of each other and that choices on different days are also independent.

a) In order to approximate $p$, 10 students were asked how many times they chose cafeteria $C_1$ during the previous 5 days. The answers were: 0, 0, 1, 1, 1, 1, 1, 2, 3, 4. For which $p$ will the empirical mean connected with the sample be equal to the expected value of the number of choices of cafeteria $C_1$?

b) Let us assume that the queueing times in cafeterias $C_1$, $C_2$ and $C_3$ (measured in minutes) are random variables from exponential distributions with parameters 1/2, 1 and 1/3, respectively. During the next 30 days, a student plans to visit cafeteria $C_1$ 15 times, cafeteria $C_2$ 5 times, and cafeteria $C_3$ 10 times. Calculate the expected value of the total amount of time (in minutes) which will be spent on queueing during those 30 days.

**4.** Mr. Smith wants to invest a part of his savings. He can open a bank deposit (in which case his profit will amount to 3 percent per year), or he can buy shares of a company, in which case his yearly profit (in percent) will be a random variable with a CDF equal to

$$F(t) = \begin{cases} 0 & \text{if } t < 1, \\ 1 - t^{-2} & \text{if } t \geq 1. \end{cases}$$

Mr. Smith has the following strategy. He tosses a coin, for which the probability of heads amounts to 1/5; if heads appear, Mr. Smith deposits his money in the bank; otherwise, he buys shares. Let $Y$ denote Mr. Smith's profit (in percent) after a year for this strategy.

a) Find the CDF of variable $Y$. Is $Y$ continuous? Is $Y$ discrete?

b) Calculate $\mathbb{E}Y$.

**5.** Let $X$ be a random variable from a distribution with density $g(x) = \frac{1}{4}a^{-4}x^3\mathbb{1}_{(0,2a]}(x)$, where $a > 0$ is a constant.

a) Find $a$, if we know that the quantile of rank $\frac{1}{16}$ for random variable $X$ is equal to 1.

b) Calculate the expected value of variable $\frac{1}{X}$ and the variance of variable $\frac{4}{X} - 5$.

**6.** Let $X$ be a random variable from a distribution such that $3\mathbb{P}(X = 1) = \mathbb{P}(X = 2) = \mathbb{P}(X = 3) = c$.

a) Find $c$.

b) Calculate $\mathbb{P}(X \leq 2 | X \geq 2)$.

c) Let $Y$ be a variable from a normal distribution with mean 1 and variance $\lambda$. Find $\lambda$, knowing that $\mathbb{E}Y^2 = \mathbb{E}X^3$.

**7.** A businessman had two conversations between 1:00 PM and 4:00 PM. The first call started at 13:00 and lasted until hour $X$, where $X$ is a random variable from a uniform distribution over the range $[13, 16]$; the other call started at hour $X$ and lasted until 16:00.

a) What is the probability that each call lasted more than an hour?

b) Calculate the mean duration of the shorter call (in hours).