Probability Calculus Midterm Test
December 5th 2014
Version A

Please choose 5 out of the 6 problems below and solve each one on a separate piece of paper. Each problem will be graded on a scale from 0 to 10 points. Duration: 120 minutes.

1. A coin was tossed 9 times, and heads appeared exactly 4 times. In this case, what is the probability that
   (a) a sequence of at least two heads (i.e. one head after the other in subsequent tosses) was obtained at least once?
   (b) a sequence of four heads appeared?

2. The price level of a barrel of oil has been observed to follow a pattern: each day, the price falls by \$1 with probability $\frac{2}{3}$, and rises by \$1 with probability $\frac{1}{3}$. The daily price movements are independent.
   (a) Calculate the probability that after five days the price will fall by \$1 with respect to the initial level.
   (b) Calculate the probability that after five days the price will rise by \$1, if we know that after two days the price level was equal to the initial level.
   (c) Are the events described in point (b), i.e.: *the price rose by \$1 after five days* and *the price level after two days was unchanged*, independent? Justify your answer!

3. A research center distributed a questionnaire to 300 small and 100 large firms, randomly chosen from a register, which was outdated by a year. According to data from the central statistical office, 0.2% of small companies and 0.5% of big companies declare bankruptcy in a given year. Using the Poisson theorem, provide
   (a) an approximation of the probability that exactly one of the surveyed firms declared bankruptcy within the last year;
   (b) an approximation of the probability that if exactly one of the surveyed firms declared bankruptcy within the last year, it was a small firm.

4. Let X be a random variable with density $g(x) = ce^{-2x}\mathbf{1}_{[0,\ln 5]}(x)$, and $Y = e^{2X}$. Find:
   (a) The constant c;
   (b) $\mathbb{P}(Y \in [1, 25])$;
   (c) The distribution of Y. Is Y continuous? Justify and if yes, provide the density function.
   (d) The expected value of Y.

5. Let X be a random variable from a uniform distribution over $[-3, 3]$, $Y = X^2$ and $Z = 3X^2 + 2$. Find:
   (a) The distribution of Y;
   (b) The expected value and the variance of Y and Z.

6. An insurance company sold 1000 yearly car insurance policies. A policy will not lead to any capital sum payments with probability equal to $\frac{1}{2}$, will lead to one payment with probability equal to $\frac{1}{3}$ and to two payments with probability equal to $\frac{1}{6}$.
   (a) Calculate the average overall number of payments from all policies of the insurance company.
   (b) Under an additional assumption that the value of a capital sum payment is a random variable with a cumulative distribution function equal to

$$F(t) = \begin{cases} 0 & t < 100 \\ \frac{t}{2000} & t \in [100, 1000) \\ 1 - \frac{1}{2}\left(1 - \frac{t}{10^4}\right)^2 & t \in [1000, 10000) \\ 1 & t \geq 10^4 \end{cases}$$

   calculate the average and the median value of a single payment, as well as the average overall payment from all policies.

Probability Calculus Midterm Test
December 5th 2014
Version B

Please choose 5 out of the 6 problems below and solve each one on a separate piece of paper. Each problem will be graded on a scale from 0 to 10 points. Duration: 120 minutes.

1. A coin was tossed 11 times, and heads appeared exactly 5 times. In this case, what is the probability that
   (a) a sequence of at least two heads (i.e. one head after the other in subsequent tosses) was obtained at least once?
   (b) a sequence of five heads appeared?

2. The price level of a barrel of oil has been observed to follow a pattern: each day, the price falls by \$1 with probability $\frac{1}{4}$, and rises by \$1 with probability $\frac{3}{4}$. The daily price movements are independent.
   (a) Calculate the probability that after five days the price will rise by \$1 with respect to the initial level.
   (b) Calculate the probability that after five days the price will fall by \$1, if we know that after two days the price level was equal to the initial level.
   (c) Are the events described in point (b), i.e.: *the price fell by \$1 after five days* and *the price level after two days was unchanged*, independent? Justify your answer!

3. A research center distributed a questionnaire to 400 small and 200 large firms, randomly chosen from a register, which was outdated by a year. According to data from the central statistical office, 0.3% of small companies and 0.4% of big companies declare bankruptcy in a given year. Using the Poisson theorem, provide
   (a) an approximation of the probability that exactly one of the surveyed firms declared bankruptcy within the last year;
   (b) an approximation of the probability that if exactly one of the surveyed firms declared bankruptcy within the last year, it was a large firm.

4. Let X be a random variable with density $g(x) = ce^{-x}\mathbf{1}_{[0,\ln 10]}(x)$, and $Y = e^X$. Find:
   (a) The constant c;
   (b) $\mathbb{P}(Y \in [1, 10])$;
   (c) The distribution of Y. Is Y continuous? Justify and if yes, provide the density function.
   (d) The expected value of Y.

5. Let X be a random variable from a uniform distribution over $[-2, 2]$, $Y = X^4$ and $Z = 2X^4 - 1$. Find:
   (a) The distribution of Y;
   (b) The expected value and the variance of Y and Z.

6. An insurance company sold 10000 yearly car insurance policies. A policy will not lead to any capital sum payments with probability equal to $\frac{3}{4}$, will lead to one payment with probability equal to $\frac{1}{8}$ and to two payments with probability equal to $\frac{1}{8}$.
   (a) Calculate the average overall number of payments from all policies of the insurance company.
   (b) Under an additional assumption that the value of a capital sum payment is a random variable with a cumulative distribution function equal to

$$F(t) = \begin{cases} 0 & t < 200 \\ \frac{t}{4000} & t \in [200, 2000) \\ 1 - \frac{2}{5}\left(1 - \frac{t}{10^4}\right)^2 & t \in [2000, 10000) \\ 1 & t \geq 10^4 \end{cases}$$

   calculate the average and the median value of a single payment, as well as the average overall payment from all policies.