

Probability Calculus Midterm Test
December 14th 2013

Please choose 7 out of the 8 problems below and solve each one on a separate piece of paper. Each problem will be graded on a scale from 0 to 10 points. Duration: 150 minutes.

1. A city has 5 east-west and 5 north-south, parallel streets (i.e. the street plan is a square grid). A driver starts at the south-east corner and drives to the north-west corner, randomly choosing one of the shortest roads (i.e. at any crossing goes either north or west). Calculate the probability that the driver will pass through the center of the city.
2. A bank lends credits to various clients. A client may be either decent (the chance he will not repay the loan is 1%), or medium-risky (the chance he will not repay the loan is 10%), or very risky (the chance he will not repay the loan is 25%). There are 50% decent clients and 25% medium-risky and very risky clients. A client approached the bank and from his credit history we know that he repayed his only loan thus far.
 - (a) What is the chance that this client is very risky?
 - (b) What is the chance that the client will not repay the next loan?
3. A transport company operates 20000 transports yearly, out of which $N_l = 12000$ are done in the summer and $N_z = 8000$ during winter. The probability of car malfunction during a transport is $p_l = 0.0001$ in the summer and $p_z = 0.0002$ during winter. Based on the Poisson theorem, approximate the probability that
 - (a) the number of transports resulting in malfunction during the summer will not exceed 3;
 - (b) the number of transports resulting in malfunction during the whole year will be greater than 2;

Assess the approximation error.

4. Let X be a random variable with a uniform distribution over $[0, 1] \cup [3, 4]$.
 - (a) Find the cumulative distribution function of X ;
 - (b) Find the quantile of rank $\frac{1}{3}$ of variable X ;
 - (c) Find the expected value and the variance of X .
5. Let X be a random variable with density $g(x) = 6xe^{-3x^2} \mathbf{1}_{(0,+\infty)}(x)$, and $Y = X^2$.
 - (a) Find the distribution of Y . Is Y continuous? Is Y discrete? Justify.
 - (b) Does X have an expected value? Does Y have an expected value? If yeas, find EY oraz $E(12 - 2Y)$.

Hint. The value of an integral of the type $\int_0^\infty e^{-ax^2} dx$ may be calculated based on the properties of the density of the normal distribution.
6. Let X be a random variable with density $g(x) = \frac{c}{\sqrt{x^5}} \mathbf{1}_{(2,+\infty)}(x)$. Find c and $P(X \leq 4 | X \geq 3)$. Does X have a variance?
7. On a field of 9m by 9m there are 81 square vegetable patches. On each patch there lives an insect (species X). In May, each insect randomly chooses a direction and moves towards one of the four borders of his patch. If there are two insects on the same border, they meet and either like each other, or fight (with probabilities $\frac{1}{2}$ each). In the latter case one of the insects dies. In the first case, they mate and each returns to his/her patch with a baby insect. Calculate the average number of insects after the mating season.
8. Let X be a random variable such that

$$P(X = 3^k) = \frac{1}{2^k}, \text{ dla } k = 1, 2, \dots,$$

and $Y = \ln X$. Do X and Y have expected values? If yes, find them.