Probability Calculus Final Exam - 05.03.2020
group A

Each problem should be solved on a separate piece of paper, you should return all 6. Each problem will be graded on a scale from 0 to 10 points. The exam result is the sum of points obtained for the 5 problems with highest grades. Please sign each piece of paper with your name and student's number and the group sign (A, B, ...). When dealing with the CDF of the standard normal distribution, please use the Φ notation, and not values from tables. Duration: 120 minutes.

1. There are three donuts and three cupcakes on a plate. A child approached the plate and took some cakes (each number of cakes, from 0 to 6, is equally probable).

a) Find the average number of donuts taken by the child.

b) Are the events "the child took as many cupcakes as donuts" and "the child took an even number of cakes" independent?

2. We know that 60% graduates of University A, 50% graduates of University B and 40% graduates of University C are competent; the rest are not.

a) During the recruitment process in company X three candidates applied, one from each university. What is the chance that exactly two of them will be competent?

b) During the recruitment process in company X three candidates applied, one from each university, and exactly one was incompetent. What is the chance that it was the graduate from university A?

c) A competent candidate has a 10% chance of being invited for an interview after submitting a CV, while an incompetent candidate – 1%. During a recruitment process in company Y, there were 300 applications in total, 100 from graduates of each of the three universities (random candidates). Using the Poisson theorem, approximate the probability that exactly one person will be invited for an interview.

3. Let X, Y and Z be independent random variables from an exponential distribution with parameter 3.

a) Find the covariance matrix for the vector $(X+Y, Y+Z)$.

b) Calculate $\mathbb{E}\left((X+Y+Z)(2X-Y+3)\right)$.

c) Calculate the value of the CDF of variable $X-Y$ at point 0.

4. Let (X,Y) be a random vector with a density $f(x,y) = c \cdot 1_{(0,1)}(x) \cdot 1_{(x^2,x)}(y)$. Find the constant c, $\mathbb{E}\left(Y^2 + X|X\right)$ and the variance of variable X.

5. We investigate tax forms of married couples. Assume that the income of a single individual is a random variable from a uniform distribution over $[1000, 4000]$. Each person pays a tax of 10% of their own income, minus 100. We assume that incomes of different individuals (including spouses) are independent.

a) Approximate the probability that among 10^6 couples, in at least 505000 the younger spouse earns more.

b) Approximate the probability that the aggregate tax paid by 10^6 couples will not exceed 299.8 mln.

6. An economy may experience periods of growth, stagnation and recession. Let us assume that a particular economy has the following properties: (a) if, in a given quarter of the year, it experiences growth, then the chance that there will also be growth in the next quarter amounts to $\frac{1}{2}$, the chance of stagnation amounts to $\frac{1}{3}$, while the chance of recession – $\frac{1}{6}$; (b) if, in a given quarter, it experiences stagnation, the chance that this will translate to the next quarter amounts to $\frac{1}{2}$, while the chances of improvement or worsening are equal; (c) if the economy is in a phase of recession, then in the next quarter all possibilities have the chance of $\frac{1}{3}$. Knowing that in the fourth quarter of 2019 the economy experienced a recession:

a) find the probability that there will be growth in the second quarter of 2020;

b) determine what is longer: the average duration (in quarters) of periods of recession in this economy or the average duration (in quarters) of periods of growth in this economy.

Probability Calculus Final Exam - 05.03.2020
group B

Each problem should be solved on a separate piece of paper, you should return all 6. Each problem will be graded on a scale from 0 to 10 points. The exam result is the sum of points obtained for the 5 problems with highest grades. Please sign each piece of paper with your name and student's number and the group sign (A, B, ...). When dealing with the CDF of the standard normal distribution, please use the Φ notation, and not values from tables. Duration: 120 minutes.

1. There are two donuts and four cupcakes on a plate. A child approached the plate and took some cakes (each number of cakes, from 0 to 6, is equally probable).

a) Find the average number of donuts taken by the child.

b) Are the events "the child took as many cupcakes as donuts" and "the child took an even number of cakes" independent?

2. We know that 60% graduates of University A, 50% graduates of University B and 40% graduates of University C are competent; the rest are not.

a) During the recruitment process in company X three candidates applied, one from each university. What is the chance that exactly two of them will be competent?

b) During the recruitment process in company X three candidates applied, one from each university, and exactly one was incompetent. What is the chance that it was the graduate from university A?

c) A competent candidate has a 10% chance of being invited for an interview after submitting a CV, while an incompetent candidate – 1%. During a recruitment process in company Y, there were 300 applications in total, 100 from graduates of each of the three universities (random candidates). Using the Poisson theorem, approximate the probability that exactly one person will be invited for an interview.

3. Let X, Y and Z be independent random variables from an exponential distribution with parameter 2.

a) Find the covariance matrix for the vector $(X - Y, Y + Z)$.

b) Calculate $\mathbb{E}\left((X + Y - Z)(X - 2Y - 2)\right)$.

c) Calculate the value of the CDF of variable $X - Z$ at point 0.

4. Let (X, Y) be a random vector with a density $f(x, y) = c \cdot 1_{(0,1)}(x) \cdot 1_{(x^2,4x)}(y)$. Find the constant c, $\mathbb{E}\left(2Y^2 - X|X\right)$ and the variance of variable X.

5. We investigate tax forms of married couples. Assume that the income of a single individual is a random variable from a uniform distribution over $[1000, 4000]$. Each person pays a tax of 10% of their own income, minus 100. We assume that incomes of different individuals (including spouses) are independent.

a) Approximate the probability that among 10^5 couples, in at least 50400 the older spouse earns more.

b) Approximate the probability that the aggregate tax paid by 10^5 couples will not exceed 30.02 mln.

6. An economy may experience periods of growth, stagnation and recession. Let us assume that a particular economy has the following properties: (a) if, in a given quarter of the year, it experiences growth, then the chance that there will also be growth in the next quarter amounts to $\frac{1}{2}$, while the chance of stagnation and recession are equal; (b) if, in a given quarter, it experiences stagnation, the chance that there will be growth in the next quarter amounts to $\frac{1}{2}$, the chance of stagnation amounts to $\frac{1}{3}$, while the chance of recession – $\frac{1}{6}$; (c) if the economy is in a phase of recession, then in the next quarter all possibilities have the chance of $\frac{1}{3}$. Knowing that in the fourth quarter of 2019 the economy experienced a recession:

a) find the probability that there will be growth in the second quarter of 2020;

b) determine what is longer: the average duration (in quarters) of periods of recession in this economy or the average duration (in quarters) of periods of growth in this economy.