Probability Calculus Final Exam - 08.03.2019
group A

Each problem should be solved on a separate piece of paper, you should return all 6. Each problem will be graded on a scale from 0 to 10 points. The exam result is the sum of points obtained for the 5 problems with highest grades. Please sign each piece of paper with your name and student's number and the group sign (A, B, ...). When dealing with the CDF of the standard normal distribution, please use the Φ notation, and not values from tables. Duration: 120 minutes.

1. There are k white balls and $10-k$ black balls in a box, where $k \in \{0, 1, 2, \ldots, 10\}$ is a random number (each possibility has the same chance). We draw five balls from the box (simultaneously).

a) Calculate the expected value of the number of white balls among those drawn.

b) Assume that $k=4$ and let's consider events A - ,,there is at least one white ball among those drawn" and B - ,,there remain at least four black balls in the box". Determine whether A and B are independent.

2. A smartphone may be manufactured in one of three facilities: 80% are manufactured in facility A, 15% in facility B and 5% in facility C. We know that one in a hundred smartphones manufactured in A, one in every fifty smartphones manufactured in B and one in every twenty smartphones manufactured in C is defective.

a) A client bought two smartphones. What is the probability that both of them are defective?

b) A client bought two smartphones and both of them were defective. What is the chance that both of them were manufactured in facility C?

c) Using the Poisson theorem, approximate the probability that in a random set of 100 smarphones there will be at most two defective ones.

3. Let X, Y, Z be independent random variables from a normal distribution with mean 1 and a variance equal to 2.

a) Find the density of random variable $X-Y$.

b) Calculate $\mathbb{E}\big[(X-2Y)^2(2Z+1)\big]$.

c) Determine whether $X+Y$ and $Y+Z$ are independent.

4. Let (X,Y) be a random vector such that: the density of X is equal to $g_X(x) = \frac{3}{8}x^2\mathbb{1}_{(0,2)}(x)$, while the density of Y conditional on X is equal to $g_{Y|X}(y|x) = \frac{1}{2x}\mathbb{1}_{\{0\leq y\leq 2x\}}$. Calculate $\mathrm{Cov}(X,Y)$ and the value of the cumulative distribution function of random variable $\frac{Y}{X}$ at point 1.

5. A courier needs to transport 200 parcels in an elevator. The weights of the packages (in kilograms) are independent random variables from a uniform distribution over the range $[2,4]$.

a) Assuming that the load capacity of the lift is 700 kg, and the courier weights 80 kg, approximate the probability that the courier will need to use the elevator twice.

b) Let's assume that parcels weighting more than 3.5 kg are oversized, and the elevator can't carry more than 40 of them at a time. Approximate the probability that the oversized packages from the delivery of 200 packages can't be loaded in the lift simultaneously.

6. A company may withhold paying an annual dividend if the financial gains are weak. If, in a given year, the dividend was paid, the chance that it will be paid in the next year amounts to 0.9; if, in a given year, the dividend was not paid, the chance that it will be paied next year amounts to 0.6.

a) Assume that the dividend in 2018 hasn't been paid. What is the chance that it will be paid in 2019 or 2020?

b) Assume that the dividend in 2018 hasn't been paid. Find the average time (in years) until the first dividend is paid.

c) Approximate the probability that a dividend will be paid in 2100.

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group C

Each problem should be solved on a separate piece of paper, you should return all 6. Each problem will be graded on a scale from 0 to 10 points. The exam result is the sum of points obtained for the 5 problems with highest grades. Please sign each piece of paper with your name and student's number and the group sign (A, B, ...). When dealing with the CDF of the standard normal distribution, please use the Φ notation, and not values from tables. Duration: 120 minutes.

1. There are k white balls and $10-k$ black balls in a box, where $k \in \{0, 1, 2, \ldots, 10\}$ is a random number (each possibility has the same chance). We draw four balls from the box (simultaneously).

a) Calculate the expected value of the number of black balls among those drawn.

b) Assume that $k = 5$ and let's consider events A - ,,there is at least one white ball among those drawn" and B - ,,there remain at least four black balls in the box". Determine whether A and B are independent.

2. A smartphone may be manufactured in one of three facilities: 90% are manufactured in facility A, 5% in facility B and 5% in facility C. We know that one in a hundred smartphones manufactured in A, one in every fifty smartphones manufactured in B and one in every twenty smartphones manufactured in C is defective.

a) A client bought two smartphones. What is the probability that both of them are defective?

b) A client bought two smartphones and both of them were defective. What is the chance that both of them were manufactured in facility B?

c) Using the Poisson theorem, approximate the probability that in a random set of 200 smarphones there will be at most two defective ones.

3. Let X, Y, Z be independent random variables from a normal distribution with mean 2 and a variance equal to 4.

a) Find the density of random variable $X - Y$.

b) Calculate $\mathbb{E}\left[(2X+Y)^2(Z-2)\right]$.

c) Determine whether $X + Y$ and $Y - Z$ are independent.

4. Let (X, Y) be a random vector such that: the density of X is equal to $g_X(x) = \frac{3}{64}x^2\mathbb{1}_{(0,4)}(x)$, while the density of Y conditional on X is equal to $g_{Y|X}(y|x) = \frac{1}{2x}\mathbb{1}_{\{0 \le y \le 2x\}}$. Calculate $\mathrm{Cov}(X, Y)$ and the value of the cumulative distribution function of random variable $\frac{Y}{X}$ at point 1.

5. A courier needs to transport 100 parcels in an elevator. The weights of the packages (in kilograms) are independent random variables from a uniform distribution over the range $[3, 7]$.

a) Assuming that the load capacity of the lift is 600 kg, and the courier weights 80 kg, approximate the probability that the courier will need to use the elevator twice.

b) Let's assume that parcels weighting more than 6 kg are oversized, and the elevator can't carry more than 30 of them at a time. Approximate the probability that the oversized packages from the delivery of 100 packages can't be loaded in the lift simultaneously.

6. A company may withhold paying an annual dividend if the financial gains are weak. If, in a given year, the dividend was paid, the chance that it will be paid in the next year amounts to 0.7; if, in a given year, the dividend was not paid, the chance that it will be paied next year amounts to 0.8.

a) Assume that the dividend in 2018 hasn't been paid. What is the chance that it will not be paid in 2019 or 2020?

b) Assume that the dividend in 2018 hasn't been paid. Find the average time (in years) until the first dividend is paid.

c) Approximate the probability that a dividend will be paid in 2100.