Probability Calculus Final Exam - 10.03.2017
group A

Each problem should be solved on a separate piece of paper, you should return all 6. Each problem will be graded on a scale from 0 to 10 points. The exam result is the sum of points obtained for the 5 problems with highest grades. Please sign each piece of paper with your name and student's number and the group sign (A, B, ...). When dealing with the CDF of the standard normal distribution, please use the Φ notation, and not values from tables. Duration: 120 minutes.

1. Five balls, numbered 1, 2, ..., 5, were randomly put in four boxes.

a) Calculate the expected value of the number of boxes having at least one ball inside.

b) What is the probability that balls numbered 1 and 2 are inside the same box, if we know that there is at least one ball in each box?

2. Let us assume that the number of cars in a family is a random variable X with distribution $\mathbb{P}(X = 0) = \mathbb{P}(X = 1) = 4/9$, $\mathbb{P}(X = 2) = 1/9$. Each car, independently from others, has been colored in one of three colors: blue, green or red (each color is equally probable).

a) Calculate the probability that among 10 randomly chosen families there are at least 2, which do not have a red car. We assume that the numbers of cars in particular families are independent.

b) Let us assume that a family does not have a red car. Calculate the probability that the family has exactly two cars.

3. Let (X, Y) be a two-dimensional random vector with density $g(x, y) = cx\mathbb{1}_{\{x\geq 0, y\geq 0, x+2y\leq 2\}}$. Calculate c, the density of X and $\mathbb{P}(\max\{X, Y\} \leq 1)$.

4. Let X, Y, Z be independent random variables, such that X and Y have normal distributions $N(1, 4)$, and Z has a normal distribuiton $N(1, 2)$.

a) Find a and b such that the variables $aX + Y$ and bZ have the same distribution.

b) Calculate $\mathbb{E}\left(\frac{X+Y}{Z^2+1}\middle| Z+1\right)$.

c) Find the covariance matrix of the vector (X, XY).

5. The probabilities that a traveller buying a train ticket will prefer to sit in a car with and without compartments are equal to 3/4 and 1/4, respectively. 400 travellers buy tickets.

a) A ticket in a train car without compartments costs \$100, and in a train car with compartments: \$110. Using the Central Limit Theorem, approximate the probability that the revenue from 400 tickets will exceed \$40800.

b) Assume that there are 400 seats in cars with compartments and 100 seats in cars without compartments. In case a seat in the preferred car is not available, the traveller buys a place in a car of the other type. Using the Central Limit Theorem, approximate the probability that each traveller will get a preferred seat (i.e. in a car with compartments or not).

6. A researcher owning 3 umbrellas walks between his home and office, taking an umbrella with him (provided there is one within reach) if it rains (which happens with probability 1/5), but not if it doesn't rain (which happens with probability 4/5). We assume that the researcher goes to work once every day. Let us introduce a Markov Chain, with states defined as the number of umbrellas within reach, regardless of whether the researcher is at home or at work.

a) Provide the transition matrix for this Markov Chain.

b) Find the stationary distribution.

c) Approximate the probability that the researcher will get wet on his way to work on a given, remote day.

Probability Calculus Final Exam - 10.03.2017
group B

Each problem should be solved on a separate piece of paper, you should return all 6. Each problem will be graded on a scale from 0 to 10 points. The exam result is the sum of points obtained for the 5 problems with highest grades. Please sign each piece of paper with your name and student's number and the group sign (A, B, ...). When dealing with the CDF of the standard normal distribution, please use the Φ notation, and not values from tables. Duration: 120 minutes.

1. Six balls, numbered 1, 2, ..., 6, were randomly put in five boxes.

a) Calculate the expected value of the number of boxes having at least one ball inside.

b) What is the probability that balls numbered 1 and 2 are inside the same box, if we know that there is at least one ball in each box?

2. Let us assume that the number of cars in a family is a random variable X with distribution $\mathbb{P}(X = 0) = \mathbb{P}(X = 1) = 3/7$, $\mathbb{P}(X = 2) = 1/7$. Each car, independently from others, has been colored in one of three colors: blue, green or red (each color is equally probable).

a) Calculate the probability that among 9 randomly chosen families there are at least 2, which do not have a red car. We assume that the numbers of cars in particular families are independent.

b) Let us assume that a family does not have a green car. Calculate the probability that the family has exactly two cars.

3. Let (X, Y) be a two-dimensional random vector with density $g(x, y) = cy\mathbb{1}_{\{x\geq 0, y\geq 0, 2x+y\leq 2\}}$. Calculate c, the density of Y and $\mathbb{P}(\max\{X, Y\} \leq 1)$.

4. Let X, Y, Z be independent random variables, such that X and Y have normal distributions $N(-1, 4)$, and Z has a normal distribution $N(1, 2)$.

a) Find a and b such that the variables $X + aY$ and bZ have the same distribution.

b) Calculate $\mathbb{E}\left(\frac{X + 2Y}{Z^2 + 2}\middle| Z + 1\right)$.

c) Find the covariance matrix of the vector (X, XZ).

5. The probabilities that a traveller buying a train ticket will prefer to sit in a car with and without compartments are equal to 3/5 and 2/5, respectively. 900 travellers buy tickets.

a) A ticket in a train car without compartments costs \$100, and in a train car with compartments: \$110. Using the Central Limit Theorem, approximate the probability that the revenue from 900 tickets will exceed \$93000.

b) Assume that there are 900 seats in cars with compartments and 100 seats in cars without compartments. In case a seat in the preferred car is not available, the traveller buys a place in a car of the other type. Using the Central Limit Theorem, approximate the probability that each traveller will get a preferred seat (i.e. in a car with compartments or not).

6. A researcher owning 3 umbrellas walks between his home and office, taking an umbrella with him (provided there is one within reach) if it rains (which happens with probability 1/4), but not if it doesn't rain (which happens with probability 3/4). We assume that the researcher goes to work once every day. Let us introduce a Markov Chain, with states defined as the number of umbrellas within reach, regardless of whether the researcher is at home or at work.

a) Provide the transition matrix for this Markov Chain.

b) Find the stationary distribution.

c) Approximate the probability that the researcher will get wet on his way to work on a given, remote day.