Probability Calculus Exam, March 14th, 2014

- The solution of a problem must include all calculations and all the steps of the reasoning, recall all theorems and formulae used, etc. A solution consisting of the final answer only will receive 0 pts.
- It is prohibited to use any notes, books, tables or calculators. Mobile phones must be switched off at all times.
- Total exam time: 150 minutes.

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1. A bank client may either be gullible (90% of clients) or skeptical (10% of clients). If a client is gullible, a bank agent will convince him to buy shares of an investment fund with probability 0.8; if the client is skeptical, with probability 0.9 he will not be convinced to invest. The investment yields profits with probability 0.1, losses with probability 0.8 and neither profits nor losses with probability 0.1. If a client does not invest, his savings remain constant. Calculate the probability that a random client will lose. (3 *pts*). We know that Mr X did not lose anything. What is the probability that he is skeptical? (3 *pts*)

2. Let X be a random variable with density $g(x) = \frac{c}{x} 1_{(1,3)}(x)$. Calculate c (2 *pts*), $\mathbb{P}(X \in (2,4))$ (2 *pts*), the distribution of the random variable $-X^4$ (3 *pts*) and $\mathbb{E}(-X^4)$ (3 *pts*).

3. Let (X, Y) be a random vector with density $g(x, y) = 5x^2 y 1_{(-1,1)}(x) 1_{(0,|x|)}(y)$. Find the covariance of random variables X and Y (7 *pts*). Verify whether X and Y are independent(3 *pts*).

4. Let (X, Y) be uniformly distributed over a triangle with vertices (0,0), (1, 2) and (2,1), i.e. with density $g(x, y) = \frac{2}{3}(1_{[0,1]}(x) 1_{[\frac{x}{2}, 2x]}(y) + 1_{(1,2]}(x) 1_{[\frac{x}{2}, 3-x]}(y))$. Find $\mathbb{E}(Y|X)$ (8 *pts*) and $\mathbb{E}(\mathbb{E}(Y|X))$ (2 *pts*).

5. Let X and Y be independent random variables distributed exponentially with parameters 2 and 4, respectively. Let $Z = XY$ and $T = X + Y - 1$. Find the expected values and variances of Z and T, and the correlation coefficient of Z and T.

6. Consider a direct selling business of cosmetics. A sales representative always tries to obtain contacts to possible new clients from clients already visited. Lest us assume that the number of new contacts provided by a single person follows a binomial distribution with parameters 2 and $\frac{1}{2}$. Let us also assume that the sales representative has a success rate of 25% (sells a product to one in four consumers).
   (a) On a given month, the representative wants to visit 10 individuals and all contacts provided by those 10 individuals. How many transactions will take place, on average? (5 *pts*)
   (b) The following month the representative wants to visit n individuals and all contacts provided by those n individuals. For which values of n will the expected number of monthly transactions exceed 20? (5 *pts*)

7. A breeding farm raises rats for medical experiments. In a single litter, a female rat breeds 4, 5, or 10 cubs with probabilities $\frac{1}{2}$, $\frac{2}{5}$, $\frac{1}{10}$, respectively. Approximate the probability that 192 female rats will breed at least 1020 cubs. (6 *pts*) Rats breed 4 times a year, and the sizes of different litters are independent. Approximate the probability that 192 female rats will breed at most 3960 cubs in total (4 *pts*).

8. A minister of finance rolls a die to decide if taxes will rise or fall. If,for a given year, taxes rose, then for the next year they will be reduced if a 5 or a 6 are obtained on the die (and rise in the remaining cases). If taxes fell, for a given year, then the next year they will fall again if only a 6 is obtained (and rise in the remaining cases). In 2013 taxes rose. Calculate the probability that a sequence of four consecutive tax increases will occur before a sequence of two consecutive decreases will take place (2013 included).

$\Phi(0,5) \approx 0,692$, $\Phi(1) \approx 0,841$, $\Phi(1,5) \approx 0,933$, $\Phi(2) \approx 0,977$, $\Phi(2,5) \approx 0,994$, $\Phi(3) \approx 0,9987$, $\Phi(4) \approx 0,99997$