Probability Calculus Exam, March 8th, 2013

- The solution of a problem must include all calculations and all the steps of the reasoning, recall all theorems and formulae used, etc. A solution consisting of the final answer only will receive 0 pts.
- It is prohibited to use any notes, books, tables or calculators. Mobile phones must be switched off at all times.
- Total exam time: 150 minutes.

1. Among ten coins, there are eight regular and two with heads on both sides. We randomly choose a coin and toss it twice; the result is two heads. What is the probability that the chosen coin has heads on both sides? (3 *pts*) We toss the same coin for a third time. What is the probability of obtaining another head? (3 *pts*)

2. The cumulative distribution function of random variable X is given by the formula $F(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{1}{9}t^2 & \text{for } t \in [0,3) \\ 1 & \text{for } t \geqslant 3 \end{cases}$
   (a) Justify that X is a continuous random variable (1 *pt*).
   (b) Find the density function of X and $Y = X^2$ (3 *pts*).
   (c) Calculate $\mathbb{E}X$, $\mathrm{Var}X$. (3 *pts*).
   (d) Is the variable $Z = \min(X, 1)$ continuous? Justify! (1 *pt*)
   (e) Calculate $\mathbb{E}Z$, where $Z = \min(X, 1)$ (2 *pts*).

3. Let $g(x, y) = \frac{1}{26}(x^2y + y^2)1_{\{0 \leqslant x \leqslant 3\}}1_{\{0 \leqslant y \leqslant 2\}}$ be the joint density function of the random vector (X, Y).
   (a) Calculate $\mathbb{P}(X \leqslant 1, Y \leqslant 1)$ (2 *pts*).
   (b) Find the marginal density functions of X and Y (3 *pts*).
   (c) Calculate $\mathbb{E}X, \mathbb{E}Y$ (3 *pts*).
   (d) Calculate the covariance of variables X and Y (2 *pts*).

4. Let $g(x, y) = 2e^{-x-y}1_{\{0<y<x\}}$ be the joint density function of variables X and Y. Calculate the marginal density functions of X and Y (3 *pts*). Verify whether X and Y are independent (justify the answer). (2 *pts*) Calculate $\mathbb{E}(X \mid Y)$ and $\mathbb{E}(X\sin(Y) + Y^2 \mid Y)$. (4 *pts*)

5. Let (X, Y) be a random vector with a normal distribution with mean $(0, 0)$ and a covariance matrix $\begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix}$.
   (a) Calculate the correlation coefficient of variables X and Y (2 *pts*).
   (b) Find the distribution of variable $X + 3Y$ (2 *pts*).
   (c) Find $a \in \mathbb{R}$ such that $X + aY$ and Y are independent. (4 *pts*)

6. A currency exchange bureau specializes in selling Swiss francs. Let us assume that the amounts bought by clients are randomly, uniformly distributed over the interval $[200, 500]$. Approximate the probability that 108 random, independent clients who will visit the bureau on March 8th, will demand at least 38250 francs (5 *pts*). What should be the amount of francs prepared by the bureau owner in the morning, in order to assure that all clients on this day will be able to buy the demanded quantity, with probability at least 0,9? (4 *pts*)

7. A stock exchange invertor has the following strategy: if, on a given day, he did not undertake any actions on the market, then the next day he will sell a block of shares with probability $\frac{1}{3}$, buy a block of shares with probability $\frac{1}{6}$, and do nothing (again) with probability $\frac{1}{2}$. If on a given day, the investor sold a block of shares, then on the next day he will sell again, buy a block of shares or refrain from any actions with probabilities $\frac{1}{3}$ each. If, on a given day, the investor bought a block of shares, then the next day he will only observe th emarket and not undertake any actions. Describe the transition matrix of the Markov Chain which corresponds to the investor's strategy (2 *pts*). Find the stationary distribution for this Markov Chain (4 *pts*). What is the approximate probability that on March, 8th the investor will not undertake any actions, if he has followed this strategy for a long time? (2 *pts*)

$\Phi(0,5) \approx 0,69$, $\Phi(0,6) \approx 0,73$, $\Phi(1) \approx 0,84$, $\Phi(1,28) \approx 0,90$, $\Phi(1,65) \approx 0,95$

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1A. Jack and Jill repetitively conduct the following experiment: both toss a coin 100 times, counting the number of tails. They then calculate the sum of tailes obtained by both. (For example, if in the 10th repetition of the experiment Jack got 52 tails and Jill got 47, then $X_{10} = 99$). Let $S_n = \frac{X_1+\ldots+X_n}{2n}$. Does the sequence $(S_n)_{n\geqslant 1}$ converge almost surely? (2 *pts*) If yes, find the limit. (4 *pts*)

2A. We randomly choose (without replacement) three numbers from the set $\{1, 2, \ldots, 10\}$. We arrange these numbers in ascending order and denote them X, Y, Z (i.e. $X < Y < Z$). Calculate $\mathbb{P}(Y = 2)$ and $\mathbb{P}(Y = 4)$ (3 *pts*). Find the conditional distribution of X when $Y = 7$ and the conditional distribution of Z when $Y = 7$ (4 *pts*). Calculate $\mathbb{E}(X + Z \mid Y)$ (3 *pts*).

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