Probability Calculus Final Exam - 05.02.2020
group A

Each problem should be solved on a separate piece of paper, you should return all 6. Each problem will be graded on a scale from 0 to 10 points. The exam result is the sum of points obtained for the 5 problems with highest grades. Please sign each piece of paper with your name and student's number and the group sign (A, B, ...). When dealing with the CDF of the standard normal distribution, please use the Φ notation, and not values from tables. Duration: 120 minutes.

1. Let (X, Y) be a random vector from a distribution with density $g(x,y) = Cx\mathbb{1}_{\{x\geq 0, x+|y|\leq 3\}}$. Calculate C, $\mathrm{Cov}(X^2, Y+5)$ and $\mathbb{P}(Y \geq 1)$.

2. Let X, Y, Z be independent random variables such that X has a normal distribution with mean 1 and variance equal to 1, Y has a normal distribution with mean 1 and variance equal to 2, and Z has a normal distribution with an unknown mean m and an unknown variance σ^2.

a) Calculate $\mathrm{Cov}(X+Y, 2X-Y)$.

b) Find $\mathbb{E}\big((X+Y)\cos(2X-Y)|2X-Y\big)$.

c) Calculate m and σ^2, if we know that variables $X+Y+Z$ and $2Z$ have the same distribution.

3. A client's level of interest may be measured by a coefficient, which we assume to be a random variable Y with a density $g_Y(y) = 2y\mathbb{1}_{(0,1)}(y)$. From statistical analyses we know that if the client's level of interest amounts to y, then X, the amount spent on purchases (in thousands of dollars), is a random variable from a uniform distribution over the range $(y, 2y)$.

a) Find the joint density of (X, Y) and the (unconditional) density of X.

b) Calculate $\mathbb{P}\big(X \geq 1|Y \leq 3/4\big)$.

4. Let us assume that the amounts tossed into a collection box by subsequent individuals during a street collection may be described by independent random variables such that X_n – the donation of the n-th individual, in dollars – has a distribution with mean 10 and variance equal to $\frac{4}{2^n}$, for $n \geq 1$.

a) Using the Chebyshev - Bienaymé inequality, provide a lower bound for the probability that the total amount donated by the first 10 individuals will fall into the range $(96, 104)$ dollars.

b) Find the limit, in terms of convergence in probability, for the average donation per individual, if we know that apart from the amounts tossed into the collection box, an additional amount of \$1000 was raised out of donations from 18 other individuals.

5. Each day, a student randomly chooses her means of transport to the university. She may either choose a tram or a bus, with probabilities 1/3 and 2/3, respectively. The duration of the tram ride, measured in minutes, is a random variable from a uniform distribution over $[20, 40]$, while the duration of the bus ride - a random variable from a uniform distribution over $[10, 50]$. We assume that trip durations and the student's decisions on various days are independent. Additionally, we know that the student will be late for classes if a ride takes more than 35 minutes.

a) Approximate the probability that the student will be late for classes at least 30 times during 100 consecutive days.

b) Approximate the probability that the student will not spend more than 49 hours in total during her travel to the university on 100 consecutive days.

6. There are two balls in a box, one of them is white and the other one is black. We perform the following infinite sequence of experiments: in each step, we randomly choose a ball and toss a symmetric coin. If heads appear, we put the ball back into the box. If tails appear, we repaint the ball (white if the ball was black and black if the ball was white) and put the repainted ball back into the box.

a) What is the probability that after two steps there will be at least one black ball in the box?

b) Find the average number of steps after which there will be two black balls in the box for the first time.

c) Approximate the probability that after 1000 steps there will be two black balls in the box.

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group B

Each problem should be solved on a separate piece of paper, you should return all 6. Each problem will be graded on a scale from 0 to 10 points. The exam result is the sum of points obtained for the 5 problems with highest grades. Please sign each piece of paper with your name and student's number and the group sign (A, B, ...). When dealing with the CDF of the standard normal distribution, please use the Φ notation, and not values from tables. Duration: 120 minutes.

1. Let (X, Y) be a random vector from a distribution with density $g(x, y) = Cy\mathbb{1}_{\{y\geq 0, |x|+y\leq 3\}}$. Calculate C, $\mathrm{Cov}(X, Y^2 + 7)$ and $\mathbb{P}(X \geq 2)$.

2. Let X, Y, Z be independent random variables such that X has a normal distribution with mean 1 and variance equal to 2, Y has a normal distribution with mean 1 and variance equal to 1, and Z has a normal distribution with an unknown mean m and an unknown variance σ^2.

a) Calculate $\mathrm{Cov}(X + Y, X - 2Y)$.

b) Find $\mathbb{E}\big((X + Y)\sin(X - 2Y)|X - 2Y\big)$.

c) Calculate m and σ^2, if we know that variables $X + Y + Z$ and $3Z$ have the same distribution.

3. A client's level of interest may be measured by a coefficient, which we assume to be a random variable Y with a density $g_Y(y) = \frac{1}{2}y\mathbb{1}_{(0,2)}(y)$. From statistical analyses we know that if the client's level of interest amounts to y, then X, the amount spent on purchases (in thousands of dollars), is a random variable from a uniform distribution over the range $(y, 2y)$.

a) Find the joint density of (X, Y) and the (unconditional) density of X.

b) Calculate $\mathbb{P}\big(X \geq 2|Y \leq 3/2\big)$.

4. Let us assume that the amounts tossed into a collection box by subsequent individuals during a street collection may be described by independent random variables such that X_n – the donation of the n-th individual, in dollars – has a distribution with mean 15 and variance equal to $\frac{4}{2^n}$, for $n \geq 1$.

a) Using the Chebyshev - Bienaymé inequality, provide a lower bound for the probability that the total amount donated by the first 10 individuals will fall into the range $(145, 155)$ dollars.

b) Find the limit, in terms of convergence in probability, for the average donation per individual, if we know that apart from the amounts tossed into the collection box, an additional amount of \$10000 was raised out of donations from 180 other individuals.

5. Each day, a student randomly chooses her means of transport to the university. She may either choose a tram or a bus, with probabilities 2/3 and 1/3, respectively. The duration of the tram ride, measured in minutes, is a random variable from a uniform distribution over $[20, 60]$, while the duration of the bus ride - a random variable from a uniform distribution over $[30, 50]$. We assume that trip durations and the student's decisions on various days are independent. Additionally, we know that the student will be late for classes if a ride takes more than 45 minutes.

a) Approximate the probability that the student will be late for classes at most 30 times during 100 consecutive days.

b) Approximate the probability that the student will spend more than 67 hours in total during her travel to the university on 100 consecutive days.

6. There are two balls in a box, one of them is white and the other one is black. We perform the following infinite sequence of experiments: in each step, we randomly choose a ball and toss a symmetric coin. If heads appear, we put the ball back into the box. If tails appear, we repaint the ball (white if the ball was black and black if the ball was white) and put the repainted ball back into the box.

a) What is the probability that after two steps there will be at least one white ball in the box?

b) Find the average number of steps after which there will be two white balls in the box for the first time.

c) Approximate the probability that after 1000 steps there will be two white balls in the box.

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group C

Each problem should be solved on a separate piece of paper, you should return all 6. Each problem will be graded on a scale from 0 to 10 points. The exam result is the sum of points obtained for the 5 problems with highest grades. Please sign each piece of paper with your name and student's number and the group sign (A, B, ...). When dealing with the CDF of the standard normal distribution, please use the Φ notation, and not values from tables. Duration: 120 minutes.

1. Let (X, Y) be a random vector from a distribution with density $g(x, y) = Cx\mathbb{1}_{\{x\geq 0, x+|y|\leq 4\}}$. Calculate C, $\mathrm{Cov}(2X^2, Y - 3)$ and $\mathbb{P}(Y \geq 2)$.

2. Let X, Y, Z be independent random variables such that X has a normal distribution with mean -1 and variance equal to 2, Y has a normal distribution with mean 1 and variance equal to 3, and Z has a normal distribution with an unknown mean m and an unknown variance σ^2.

a) Calculate $\mathrm{Cov}(3X + Y, X - 2Y)$.

b) Find $\mathbb{E}\big((3X + Y)\cos(X - 2Y)|X - 2Y\big)$.

c) Calculate m and σ^2, if we know that variables $X + Y + Z$ and $-2Z$ have the same distribution.

3. A client's level of interest may be measured by a coefficient, which we assume to be a random variable Y with a density $g_Y(y) = 2y\mathbb{1}_{(0,1)}(y)$. From statistical analyses we know that if the client's level of interest amounts to y, then X, the amount spent on purchases (in thousands of dollars), is a random variable from a uniform distribution over the range $(y, 3y)$.

a) Find the joint density of (X, Y) and the (unconditional) density of X.

b) Calculate $\mathbb{P}\big(X \geq 1|Y \leq 2/3\big)$.

4. Let us assume that the amounts tossed into a collection box by subsequent individuals during a street collection may be described by independent random variables such that X_n – the donation of the n-th individual, in dollars – has a distribution with mean 15 and variance equal to $\frac{4}{2^n}$, for $n \geq 1$.

a) Using the Chebyshev - Bienaymé inequality, provide a lower bound for the probability that the total amount donated by the first 15 individuals will fall into the range $(220, 230)$ dollars.

b) Find the limit, in terms of convergence in probability, for the average donation per individual, if we know that apart from the amounts tossed into the collection box, an additional amount of \$500 was raised out of donations from 25 other individuals.

5. Each day, a student randomly chooses her means of transport to the university. She may either choose a tram or a bus, with probabilities 2/3 and 1/3, respectively. The duration of the tram ride, measured in minutes, is a random variable from a uniform distribution over $[10, 50]$, while the duration of the bus ride - a random variable from a uniform distribution over $[20, 40]$. We assume that trip durations and the student's decisions on various days are independent. Additionally, we know that the student will be late for classes if a ride takes more than 35 minutes.

a) Approximate the probability that the student will be late for classes at most 35 times during 100 consecutive days.

b) Approximate the probability that the student will spend more than 51 hours in total during her travel to the university on 100 consecutive days.

6. There are two balls in a box, one of them is white and the other one is black. We perform the following infinite sequence of experiments: in each step, we randomly choose a ball and toss a symmetric coin. If heads appear, we put the ball back into the box. If tails appear, we repaint the ball (white if the ball was black and black if the ball was white) and put the repainted ball back into the box.

a) What is the probability that after two steps there will be a white ball and a black ball in the box?

b) Find the average number of steps after which there will be two black balls in the box for the first time.

c) Approximate the probability that after 1000 steps there will be at least one white ball in the box.

Probability Calculus Final Exam - 05.02.2020
group D

Each problem should be solved on a separate piece of paper, you should return all 6. Each problem will be graded on a scale from 0 to 10 points. The exam result is the sum of points obtained for the 5 problems with highest grades. Please sign each piece of paper with your name and student's number and the group sign (A, B, ...). When dealing with the CDF of the standard normal distribution, please use the Φ notation, and not values from tables. Duration: 120 minutes.

1. Let (X, Y) be a random vector from a distribution with density $g(x, y) = Cy\mathbb{1}_{\{|x|+y\leq 4,\, y\geq 0\}}$. Calculate C, $\mathrm{Cov}(X + 2, 2Y^2)$ and $\mathbb{P}(X \geq 3)$.

2. Let X, Y, Z be independent random variables such that X has a normal distribution with mean 1 and variance equal to 3, Y has a normal distribution with mean -2 and variance equal to 1, and Z has a normal distribution with an unknown mean m and an unknown variance σ^2.

a) Calculate $\mathrm{Cov}(X + Y, X - 3Y)$.

b) Find $\mathbb{E}\big((X + Y)\sin(X - 3Y)|X - 3Y\big)$.

c) Calculate m and σ^2, if we know that variables $X + 2Y + Z$ and $2Z$ have the same distribution.

3. A client's level of interest may be measured by a coefficient, which we assume to be a random variable Y with a density $g_Y(y) = \frac{1}{2}y\mathbb{1}_{(0,2)}(y)$. From statistical analyses we know that if the client's level of interest amounts to y, then X, the amount spent on purchases (in thousands of dollars), is a random variable from a uniform distribution over the range $(y, 3y)$.

a) Find the joint density of (X, Y) and the (unconditional) density of X.

b) Calculate $\mathbb{P}\big(X \geq 1|Y \leq 1\big)$.

4. Let us assume that the amounts tossed into a collection box by subsequent individuals during a street collection may be described by independent random variables such that X_n – the donation of the n-th individual, in dollars – has a distribution with mean 10 and variance equal to $\frac{4}{2^n}$, for $n \geq 1$.

a) Using the Chebyshev - Bienaymé inequality, provide a lower bound for the probability that the total amount donated by the first 15 individuals will fall into the range $(146, 154)$ dollars.

b) Find the limit, in terms of convergence in probability, for the average donation per individual, if we know that apart from the amounts tossed into the collection box, an additional amount of \$1000 was raised out of donations from 15 other individuals.

5. Each day, a student randomly chooses her means of transport to the university. She may either choose a tram or a bus, with probabilities 1/3 and 2/3, respectively. The duration of the tram ride, measured in minutes, is a random variable from a uniform distribution over $[30, 50]$, while the duration of the bus ride - a random variable from a uniform distribution over $[20, 60]$. We assume that trip durations and the student's decisions on various days are independent. Additionally, we know that the student will be late for classes if a ride takes more than 45 minutes.

a) Approximate the probability that the student will be late for classes at least 35 times during 100 consecutive days.

b) Approximate the probability that the student will not spend more than 66 hours in total during her travel to the university on 100 consecutive days.

6. There are two balls in a box, one of them is white and the other one is black. We perform the following infinite sequence of experiments: in each step, we randomly choose a ball and toss a symmetric coin. If heads appear, we put the ball back into the box. If tails appear, we repaint the ball (white if the ball was black and black if the ball was white) and put the repainted ball back into the box.

a) What is the probability that after two steps there will be only white balls in the box?

b) Find the average number of steps after which there will be two white balls in the box for the first time.

c) Approximate the probability that after 1000 steps there will be two white balls or two black balls in the box.