Probability Calculus Final Exam - 08.02.2019
group A

Each problem should be solved on a separate piece of paper, you should return all 6. Each problem will be graded on a scale from 0 to 10 points. The exam result is the sum of points obtained for the 5 problems with highest grades. Please sign each piece of paper with your name and student's number and the group sign (A, B, ...). When dealing with the CDF of the standard normal distribution, please use the Φ notation, and not values from tables. Duration: 120 minutes.

1. Let (X, Y) be a two-dimensional vector with density $g(x, y) = 60x^2y\mathbb{1}_{\{0\le x\le 2y\le 1\}}$. Calculate the density of variable Y, $\mathrm{Cov}(X, Y^2)$ and $\mathbb{E}(2XY - 5Y^2 + 1|Y)$.

2. Let X, Y be independent, identically distributed random variables from a normal distribution with mean 1 and variance equal to 4.

a) Find the covariance matrix of the vector $(X - 3, X + Y)$.

b) What does $\mathbb{P}(X + Y \le 2) + \mathbb{E}XY$ amount to? Justify your answer.

c) Determine whether variables $-5X + 12$ and $3X + 4Y$ have the same distribution.

3. A user downloads a certain amount of unrelated files from the internet. The amount of time needed to download a single file (measured in minutes) is a random variable from a uniform distribution over the interval $[0, 20]$ and depends only on the downloaded file. If the download time is longer than 19 minutes, the server breaks the connection and moves to downloading the next file on the list.

a) Using the Chebyshev-Bienaymé inequality, assess the probability that upon downloading 40 files, the server will not break the connection at all or will break the connection at least 4 times.

b) Let X_n denote the amount of time needed to download the n-th file from the list (we assume $X_n = 19$ if the server breaks the connection). Find the limit, in terms of almost sure convergence, of the sequence

$$\frac{(X_1 + X_2 + \ldots + X_n)^2}{5n^2 + 1}, \qquad n = 1, 2, \ldots.$$

4. A meteorologist studies te atmosphere dynamics over Sicily in July. Let us denote the aggregate amount of rainfall (in liters per square meter) by X, and by Y – the average temperature level (in degrees Celsius). From historical data we know that random variable Y has a uniform distribution over the interval $[20, 30]$; moreover, if $Y = y$, the random variable X has a uniform distribution over the range $[0, y/10]$.

a) Calculate the probability that the amount od rainfall in July next year will not surpass $\frac{1}{2}$ liter/m^2.

b) Calculate the expected value of the temperature level under the assumption that the amount of rainfall will be equal to $\frac{1}{2}$ liter/m^2.

5. A Bank expects that on February 11th there will be 100 clients wanting to withdraw cash. Each of these individuals will either withdraw a certain amount in PLN, in USD or in EUR (the probabilities of these possibilities amount to 3/5, 1/5 and 1/5, respectively).

a) Using the de Moivre-Laplace Theorem or the Central Limit Theorem, approximate the probability that the number of individuals interested in withdrawing in PLN or EUR will be larger than the number of individuals withdrawing in USD by at least 62.

b) Let us further assume that a single withdrawal in PLN (in thousands of PLN) has an exponential distribution with parameter 1. Using the Central Limit Theorem, approximate the probability that the aggregate amount of PLN withdrawn on February 11th will exceed 65 thousand.

Hint for b): the amount withdrawn in PLN for each of the 100 individuals may be expressed as $X \cdot Y$, where X, Y are independent random variables such that $X \sim$Exp(1) and $\mathbb{P}(Y = 1) = \frac{3}{5}$, $\mathbb{P}(Y = 0) = \frac{2}{5}$.

6. Mr Smith keeps two books borrowed from the library on his shelf. Each Monday, he randomly chooses one of these books and reads it during the week; on Saturday, he returns this book to the library and randomly chooses a new book, which he puts on his shelf. A book borrowed form the library may either be "thin" (less than 100 pages) or "fat" (at least 100 pages) – the corresponding probabilities amount to $\frac{3}{5}$ and $\frac{2}{5}$. On Monday, February 4th, 2019, Mr Smith had two fat books on his shelf.

a) What is the probability that on February 18th, 2019 there will be two fat books on the shelf?

b) After how many weeks, on average, will there be two thin books on the shelf for the first time?

c) What is the approximate probability that after 100 weeks there will be one fat book and one thin book on the shelf?

Hint: the states of the corresponding Markov Chain may be described by the types of books on the shelf at the beginning of the week, eg. TT (two thin books), TF (one thin and one fat book), etc.

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group C

Each problem should be solved on a separate piece of paper, you should return all 6. Each problem will be graded on a scale from 0 to 10 points. The exam result is the sum of points obtained for the 5 problems with highest grades. Please sign each piece of paper with your name and student's number and the group sign (A, B, ...). When dealing with the CDF of the standard normal distribution, please use the Φ notation, and not values from tables. Duration: 120 minutes.

1. Let (X, Y) be a two-dimensional vector with density $g(x, y) = 80xy^2 \mathbb{1}_{\{0 \leq x \leq 2y \leq 1\}}$. Calculate the density of variable Y, $\mathrm{Cov}(X, Y^2)$ and $\mathbb{E}(-3XY + 2Y^2 - 10|Y)$.

2. Let X, Y be independent, identically distributed random variables from a normal distribution with mean -2 and variance equal to 2.

a) Find the covariance matrix of the vector $(2X - 1, X - Y)$.

b) What does $\mathbb{P}(X + Y \leq -4) - \mathbb{E}XY$ amount to? Justify your answer.

c) Determine whether variables $-5X - 12$ and $-3X + 4Y$ have the same distribution.

3. A user downloads a certain amount of unrelated files from the internet. The amount of time needed to download a single file (measured in minutes) is a random variable from a uniform distribution over the interval $[0, 30]$ and depends only on the downloaded file. If the download time is longer than 28 minutes, the server breaks the connection and moves to downloading the next file on the list.

a) Using the Chebyshev-Bienaymé inequality, assess the probability that upon downloading 30 files, the server will not break the connection at all or will break the connection at least 4 times.

b) Let X_n denote the amount of time needed to download the n-th file from the list (we assume $X_n = 28$ if the server breaks the connection). Find the limit, in terms of almost sure convergence, of the sequence

$$\frac{(X_1 + X_2 + \ldots + X_n)^2}{4n^2 - 3n}, \qquad n = 1, 2, \ldots.$$

4. A meteorologist studies te atmosphere dynamics over Sicily in July. Let us denote the aggregate amount of rainfall (in liters per square meter) by X, and by Y – the average temperature level (in degrees Celsius). From historical data we know that random variable Y has a uniform distribution over the interval $[30, 40]$; moreover, if $Y = y$, the random variable X has a uniform distribution over the range $[0, y/10]$.

a) Calculate the probability that the amount od rainfall in July next year will not surpass 2 liter/m^2.

b) Calculate the expected value of the temperature level under the assumption that the amount of rainfall will be equal to 2 liter/m^2.

5. A Bank expects that on February 11th there will be 100 clients wanting to withdraw cash. Each of these individuals will either withdraw a certain amount in PLN, in USD or in EUR (the probabilities of these possibilities amount to 2/5, 1/5 and 2/5, respectively).

a) Using the de Moivre-Laplace Theorem or the Central Limit Theorem, approximate the probability that the number of individuals interested in withdrawing in PLN or EUR will be larger than the number of individuals withdrawing in USD by at least 22.

b) Let us further assume that a single withdrawal in PLN (in thousands of PLN) has an exponential distribution with parameter 1/2. Using the Central Limit Theorem, approximate the probability that the aggregate amount of PLN withdrawn on February 11th will exceed 78 thousand.

Hint for b): the amount withdrawn in PLN for each of the 100 individuals may be expressed as $X \cdot Y$, *where* X, Y *are independent random variables such that* $X \sim Exp(\frac{1}{2})$ *and* $\mathbb{P}(Y = 1) = \frac{2}{5}$, $\mathbb{P}(Y = 0) = \frac{3}{5}$.

6. Mr Smith keeps two books borrowed from the library on his shelf. Each Monday, he randomly chooses one of these books and reads it during the week; on Saturday, he returns this book to the library and randomly chooses a new book, which he puts on his shelf. A book borrowed form the library may either be "thin" (less than 100 pages) or "fat" (at least 100 pages) – the corresponding probabilities amount to $\frac{4}{5}$ and $\frac{1}{5}$. On Monday, February 4th, 2019, Mr Smith had two fat books on his shelf.

a) What is the probability that on February 18th, 2019 there will be two fat books on the shelf?

b) After how many weeks, on average, will there be two thin books on the shelf for the first time?

c) What is the approximate probability that after 100 weeks there will be one fat book and one thin book on the shelf?

Hint: the states of the corresponding Markov Chain may be described by the types of books on the shelf at the beginning of the week, eg. TT (two thin books), TF (one thin and one fat book), etc.