Probability Calculus Final Exam - 05.02.2018
group A

Each problem should be solved on a separate piece of paper, you should return all 6. Each problem will be graded on a scale from 0 to 10 points. The exam result is the sum of points obtained for the 5 problems with highest grades. Please sign each piece of paper with your name and student's number and the group sign (A, B, ...). When dealing with the CDF of the standard normal distribution, please use the Φ notation, and not values from tables. Duration: 120 minutes.

1. Let X, Y be independent random variables with densities $g_X(x) = 2x\mathbb{1}_{[0,1]}(x)$ and $g_Y(y) = \frac{3}{8}y^2\mathbb{1}_{[0,2]}(y)$, respectively. Calculate $\mathrm{Cov}(X, XY + 4)$, $\mathbb{P}(X + Y \leq 1)$ and $\mathbb{E}(YX^2 + \sin Y \mid Y)$.

2. Let (X, Y) be a two-dimensional normal random vector with mean $(2, -1)$ and a covariance matrix equal to $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$.

a) Verify whether variables $X - Y + 3$ and $3X + 6Y - 12$ are independent.

b) Does there exist a value a such that variables $aX + Y$ and $X + 2aY$ have the same distribution?

3. We roll two regular dice and note the maximum number obtained on a sheet of paper. We repeat the procedure infinitely many times.

a) Using the Chebyshev-Bienaymé inequality, assess the probability that the number of sixes among the first 324 numbers noted is at least 109 or at most 89.

b) Let X_n be the n-th number noted. Verify whether the series $((X_1 + X_2 + \ldots + X_{2n})/n)^3$, $n = 1, 2, \ldots$ converges a.s.. If yes, find the limit.

4. There is a measurement device in a magazine. Due to lack of documentation and information on its usage, it is only known that the (unknown) accuracy X of this device is a random variable from an exponential distribution with parameter 5. The error Y connected with the usage of the device, assuming $X = x$, is a uniform random variable over the interval $(0, e^{-5x})$.

a) Calculate the probability that the error connected with the usage of the device will exceed $1/2$.

b) Find $\mathbb{E}(X|Y)$.

5. The national debt of a certain country, in millions of crowns, grows each day by an independent, random amount described by a uniform distribution over the interval $[0, 1]$. Each day, the government may intervene (which happens with probability $1/9$); as a result of the intervention, the debt on a given day does not increase. We assume that the decision to intervene is made based on external conditions independent on the debt behavior and previous interventions.

a) Approximate the probability that during the first 162 days the government will intervene at least 20 times.

b) Approximate the probability that during the first 288 days the debt will grow by less than 124 millions of crowns.

Hint for b): the debt increase on a given day may be presented as $X \cdot Y$, where X and Y are independent random variables such that X has a uniform distribution over $[0, 1]$ and $\mathbb{P}(Y = 1) = \frac{8}{9}$, $\mathbb{P}(Y = 0) = \frac{1}{9}$.

6. An insurance company offering short term monthly medical insurance introduced the following system of discounts. Each client benefits from a discount of 0%, 10%, 20% or 30%. If during a given month there are no insurance claims, the discount for the next month grows by 10 percentage points (until reaching the maximum level of 30%); if there is a claim, the discount for the next month diminishes by 10 percentage points (until reaching the minimum level of 0%). Mr. Smith decides to buy these short term insurance policies. Initially, he is awarded a discount of 10%. We know that the probability that during a given month Mr. Smith will need to use medical services amounts to $\frac{1}{3}$.

a) What is the probability that after three months Mr. Smith's discount will be equal to 0%?

b) Calculate the expected number of months until Mr. Smith's discount reaches 30% for the first time.

c) Approximate the probability that in the 100th month Mr. Smith's discount will amount to 20% or 30%.

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group D

Each problem should be solved on a separate piece of paper, you should return all 6. Each problem will be graded on a scale from 0 to 10 points. The exam result is the sum of points obtained for the 5 problems with highest grades. Please sign each piece of paper with your name and student's number and the group sign (A, B, ...). When dealing with the CDF of the standard normal distribution, please use the Φ notation, and not values from tables. Duration: 120 minutes.

1. Let X, Y be independent random variables with densities $g_X(x) = 2x\mathbb{1}_{[0,1]}(x)$ and $g_Y(y) = 3y^2\mathbb{1}_{[0,1]}(y)$, respectively. Calculate $\mathrm{Cov}(Y, XY + 2)$, $\mathbb{P}(X + Y \leq 1)$ and $\mathbb{E}(XY^2 - \cos X \mid X)$.

2. Let (X, Y) be a two-dimensional normal random vector with mean $(-1, 2)$ and a covariance matrix equal to $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$.

a) Verify whether variables $X + Y + 3$ and $4X - 3Y + 13$ are independent.

b) Does there exist a value a such that variables $X + aY$ and $2aX + Y$ have the same distribution?

3. We roll two regular dice and note the minimum number obtained on a sheet of paper. We repeat the procedure infinitely many times.

a) Using the Chebyshev-Bienaymé inequality, assess the probability that the number of twos among the first 200 numbers noted is at least 60 or at most 40.

b) Let X_n be the n-th number noted. Verify whether the series $((X_1 + X_2 + \ldots + X_{3n})/n)^2$, $n = 1, 2, \ldots$ converges a.s.. If yes, find the limit.

4. There is a measurement device in a magazine. Due to lack of documentation and information on its usage, it is only known that the (unknown) accuracy X of this device is a random variable from an exponential distribution with parameter 4. The error Y connected with the usage of the device, assuming $X = x$, is a uniform random variable over the interval $(0, e^{-4x})$.

a) Calculate the probability that the error connected with the usage of the device will exceed $1/5$.

b) Find $\mathbb{E}(X|Y)$.

5. The national debt of a certain country, in millions of crowns, grows each day by an independent, random amount described by a uniform distribution over the interval $[0, 2]$. Each day, the government may intervene (which happens with probability $1/6$); as a result of the intervention, the debt on a given day does not increase. We assume that the decision to intervene is made based on external conditions independent on the debt behavior and previous interventions.

a) Approximate the probability that during the first 180 days the government will intervene at least 35 times.

b) Approximate the probability that during the first 240 days the debt will grow by less than 190 millions of crowns.

Hint for b): the debt increase on a given day may be presented as $X \cdot Y$*, where* X *and* Y *are independent random variables such that* X *has a uniform distribution over* $[0, 2]$ *and* $\mathbb{P}(Y = 1) = \frac{5}{6}$, $\mathbb{P}(Y = 0) = \frac{1}{6}$.

6. An insurance company offering short term monthly medical insurance introduced the following system of discounts. Each client benefits from a discount of 0%, 10%, 20% or 30%. If during a given month there are no insurance claims, the discount for the next month grows by 10 percentage points (until reaching the maximum level of 30%); if there is a claim, the discount for the next month diminishes by 10 percentage points (until reaching the minimum level of 0%). Mr. Smith decides to buy these short term insurance policies. Initially, he is awarded a discount of 10%. We know that the probability that during a given month Mr. Smith will need to use medical services amounts to $\frac{2}{3}$.

a) What is the probability that after three months Mr. Smith's discount will be equal to 10%?

b) Calculate the expected number of months until Mr. Smith's discount reaches 0% for the first time.

c) Approximate the probability that in the 100th month Mr. Smith's discount will amount to 0% or 30%.