Probability Calculus Final Exam - 06.02.2017
group A

Each problem should be solved on a separate piece of paper, you should return all 6. Each problem will be graded on a scale from 0 to 10 points. The exam result is the sum of points obtained for the 5 problems with highest grades. Please sign each piece of paper with your name and student's number and the group sign (A, B, ...). When dealing with the CDF of the standard normal distribution, please use the Φ notation, and not values from tables. Duration: 120 minutes.

1. Let (X, Y) be a two-dimensional random vector with density $g(x, y) = \frac{3}{8}x\mathbb{1}_{\{0<x\leq y\leq 2x\leq 4\}}$. Calculate $\mathbb{E}X$, $\mathrm{Cov}(X, Y)$ and $\mathbb{P}(Y \leq 1)$.

2. Let X, Y be independent random variables from normal distributions with means equal to -1 and variances equal to 1 and 2, respectively.

a) Find the distribution of the variable $2X - Y + 2$.

b) Find the covariance matrix of the vector $(2X - Y + 2, X + Y)$.

c) Calculate $\mathbb{E}\big((2X - Y + 2)^2 + (X + Y)^2 | X + Y\big)$.

3. Let us assume that relative air humidity is a random variable from a distribution with density $g(x) = \frac{1}{x \ln 2}\mathbb{1}_{[50,100]}(x)$. If the humidity is x, the strength of a GSM signal emitted by a transmitter is a random variable from an exponential distribution with parameter $x/10$. Let X, Y denote the humidity and signal strength today.

a) Find the density of the vector (X, Y) and of the variable Y.

b) Calculate $\mathbb{P}(X \geq 75|Y)$.

4. The amounts of money allocated to travel by citizens of country C_1 are independent random variables X_1, X_2, ... from an exponential distribution with parameter $1/2$. The amounts of money allocated to travel by citizens of country C_2 are independent random variables Y_1, Y_2, ... (independent also from X_1, X_2, ...) from an exponential distribution with parameter $1/3$.

a) Using the Chebyshev-Bienaymé inequality, assess the probability that the amount of money allocated to travel by 30 chosen individuals from country C_1 differs from the amount allocated to travel by 20 chosen individuals from country C_2 by at least 50.

b) Does the sequence $\frac{X_1+Y_1+X_2+Y_2+\ldots+X_n+Y_n}{2n+5}$, $n = 1, 2, \ldots$ converge almost surely? If yes, find the limit.

5. 600 clients approach a kiosk which sells lottery tickets. Each client, independently from others, buys one, two or three tickets (each of these numbers is equally probable).

a) Approximate the probability that the kiosk will sell more than 1220 tickets.

b) Each ticket costs \$10, on average every tenth client pays by credit card. Approximate the probability that the amount of cash obtained from all the purchases will be smaller than \$11000.

6. Each day, Mr. Smith goes shopping to one of three shops: S_1, S_2 or S_3. If on a given day Mr. Smith obtains a discount in the shop of his choice, he shops there also the next day; if he does not obtain a discount, he chooses one of the other shops (with equal probabilities). The probabilities of obtaining a discount in S_1, S_2 and S_3 are equal to $1/2$, $1/3$ and $1/5$, respectively. On the first day, Mr. Smith went to S_1.

a) Calculate the probability that on the third day Mr. Smith went to S_3.

b) Approximate the probability that on the 100th day Mr. Smith went to S_3.

c) Approximate the probability that on the 100th day Mr. Smith obtained a discount.

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group B

Each problem should be solved on a separate piece of paper, you should return all 6. Each problem will be graded on a scale from 0 to 10 points. The exam result is the sum of points obtained for the 5 problems with highest grades. Please sign each piece of paper with your name and student's number and the group sign (A, B, ...). When dealing with the CDF of the standard normal distribution, please use the Φ notation, and not values from tables. Duration: 120 minutes.

1. Let (X, Y) be a two-dimensional random vector with density $g(x, y) = \frac{3}{16} x \mathbb{1}_{\{0<x\leq y\leq 3x\leq 6\}}$. Calculate $\mathbb{E}X$, $\mathrm{Cov}(X, Y)$ and $\mathbb{P}(Y \leq 2)$.

2. Let X, Y be independent random variables from normal distributions with means equal to -2 and variances equal to 2 and 1, respectively.

a) Find the distribution of the variable $X - 2Y + 1$.

b) Find the covariance matrix of the vector $(X - 2Y + 1, X + Y)$.

c) Calculate $\mathbb{E}\big((X + Y)^2 + (X - 2Y + 1)^2 | X - 2Y + 1\big)$.

3. Let us assume that relative air humidity is a random variable from a distribution with density $g(x) = \frac{1}{x \ln 2} \mathbb{1}_{[40,80]}(x)$. If the humidity is x, the strength of a GSM signal emitted by a transmitter is a random variable from an exponential distribution with parameter $x/20$. Let X, Y denote the humidity and signal strength today.

a) Find the density of the vector (X, Y) and of the variable Y.

b) Calculate $\mathbb{P}(X \geq 60|Y)$.

4. The amounts of money allocated to travel by citizens of country C_1 are independent random variables X_1, X_2, ... from an exponential distribution with parameter $1/3$. The amounts of money allocated to travel by citizens of country C_2 are independent random variables Y_1, Y_2, ... (independent also from X_1, X_2, ...) from an exponential distribution with parameter $1/2$.

a) Using the Chebyshev-Bienaymé inequality, assess the probability that the amount of money allocated to travel by 20 chosen individuals from country C_1 differs from the amount allocated to travel by 30 chosen individuals from country C_2 by at least 40.

b) Does the sequence $\frac{X_1+Y_1+X_2+Y_2+\ldots+X_n+Y_n}{3n-1}$, $n = 1, 2, \ldots$ converge almost surely? If yes, find the limit.

5. 360 clients approach a kiosk which sells lottery tickets. Each client, independently from others, buys one, three or four tickets (each of these numbers is equally probable).

a) Approximate the probability that the kiosk will sell more than 1000 tickets.

b) Each ticket costs \$10, on average every fourth client pays by credit card. Approximate the probability that the amount of cash obtained from all the purchases will be smaller than \$7230.

6. Each day, Mr. Smith goes shopping to one of three shops: S_1, S_2 or S_3. If on a given day Mr. Smith obtains a discount in the shop of his choice, he shops there also the next day; if he does not obtain a discount, he chooses one of the other shops (with equal probabilities). The probabilities of obtaining a discount in S_1, S_2 and S_3 are equal to $1/2$, $1/5$ and $1/7$, respectively. On the first day, Mr. Smith went to S_1.

a) Calculate the probability that on the third day Mr. Smith went to S_2.

b) Approximate the probability that on the 100th day Mr. Smith went to S_2.

c) Approximate the probability that on the 100th day Mr. Smith obtained a discount.