Probability Calculus Final Exam - 03.02.2016
group A

Each problem should be solved on a separate piece of paper, you should return all 6. Solving 5 out of 6 problems correctly will give you the maximum number of points for the exam. Each problem will be graded on a scale from 0 to 10 points. Please sign each piece of paper with your name and student's number. Duration: 120 minutes.

1. Let U_1 and U_2 be independent random variables from a uniform distribution over $[0,2]$. Let $X = U_1$ and $Y = \max\{U_1, U_2\}$.
   (a) Find the joint distribution of (X, Y).
   (b) Calculate $\mathbb{E}Y$ and $\mathrm{Cov}(X, Y)$.
   (c) Calculate $\mathbf{P}(X \geqslant Y)$.

2. Let X, Y and Z be random variables from an exponential distribution with parameter $\frac{1}{2}$. Assume that the pairs X and Y as well as Y and Z are independent, but Z is correlated with X: $\mathrm{Cov}(X, Z) = 1$.
   (a) What is the distribution of $X + Y$?
   (b) Calculate the correlation coefficient of the variables $X + Y$ and Z.
   (c) Knowing that $\mathrm{Cov}(XZ, Y) = 0$ find the variance of the random variable $X \cdot Y + Z$.

3. Let (X, Y) be a random vector with density $f(x, y)$ such that
$$f(x, y) = c \cdot \mathbf{1}_{(0,4-x^2)}(y) \cdot \mathbf{1}_{(-2,2)}(x).$$
   (a) Calculate the constant c.
   (b) Find $\mathbb{E}(Y|X)$.
   (c) Are X and Y independent? Justify your answer!

4. Let $X_1, X_2, \ldots$ be random variables describing the sums (in dollars) donated by subsequent individuals during a street collection organized by an NGO. Assume that the amounts donated are independent, but that the individuals become more generous as time goes by: $\mathbb{E}X_n = 2n$, while $\mathrm{Var}X_n = 1$. (a) Using the Chebyshev-Bienaymé inequality find an upper bound to the probability that the first three individuals will jointly donate an amount which differs by at least 6 dollars from the amount donated by the sixth individual. (b) Find constants a and b such that for large n the expression $an+b$ is the best possible approximation of the average amount donated by the first n individuals, i.e. such that $\frac{X_1+\ldots+X_n}{n} - (an+b) \xrightarrow{\mathbf{P}} 0$.

5. In American roulette, 18 pockets are red, 18 are black, and 2 are green. We can bet x dollars that the ball will land in one of the red pockets or that it will land in one of the black pockets. If the ball lands in a pocket of our choice, we win x dollars; otherwise we lose the bet amount. When entering the casino we have 1444 dollars. Approximate the probability that after 1444 bets of 1 dollar we will be at least as wealthy as when entering the casino. Calculate the analogous probability for the strategy of betting 4 dollars 361 times. Which strategy is better? Justify your answer.

6. Let us assume that changes of the president of one of the governmental companies occur in the following manner. There are four experts who can be elected for the position: two of them (A_1, A_2) support party A, and the other two (B_1, B_2) support party B. If, during a given quarter of the year, there are no public elections, the prime minister (from the party governing at the time) decides whether the current company president remains in position or is substituted by a different expert from the same political camp (with probability $\frac{1}{2}$). However, in each quarter, with probability $\frac{1}{12}$, public elections leading to a change of the governing party may be held; in this case, the company president is immediately substituted with expert number 1 from the other political camp. Find:
   (a) the probability that expert A_1, who chaired the company in the first quarter of 2016, will remain in position for the next three quarters;
   (b) the probability that if this mechanism has been functioning for years, expert A_1 will be chairman in the first quarter of 2016;
   (c) the average time (in quarters) since the first quarter of 2016, when A_1 is in charge of the company, until B_1 takes over.

$\Phi(1) \approx 0{,}841$, $\Phi(1{,}5) \approx 0{,}933$, $\Phi(2) \approx 0{,}977$, $\Phi(2{,}5) \approx 0{,}994$, $\Phi(3) \approx 0{,}9987$, $\Phi(4) \approx 0{,}99997$