Probability Calculus Final Exam - 30.01.2015
grupa A

Each problem should be solved on a separate piece of paper, you should return all 6. Solving 5 out of 6 problems correctly will give you the maximum number of points for the exam. Each problem will be graded on a scale from 0 to 10 points. Please sign each piece of paper with your name and student's number. Duration: 120 minutes.

1. Let (X, Y) be a random variable with density $g(x, y) = \frac{1}{2x}\mathbb{1}_{\{0<y\leqslant x\leqslant 2\}}$.

   (a) Find $\mathbb{E}X$, $\mathbb{E}Y$ and $\mathrm{Cov}(X, Y)$.

   (b) Find $\mathbb{P}(X \geqslant Y)$.

2. Let (X, Y) be a random vector from a normal distribution with mean $(0, 0)$ and a covariance matrix equal to $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

   (a) Calculate the variance of random variable $X - Y$.

   (b) Are Y^2 and $(X - Y)^2$ independent? Justify your answer!

   (c) What is the distribution of the random variable $Z = Y^2 + (X - Y)^2$? Provide the density or the name of this distribution.

   (d) Calculate $\mathbb{E}Z$ and $\mathrm{Var}Z$.

3. An analyst observed that the daily change of a stock market index, expressed in percent, is a random variable with density $g(x) = \frac{1}{100}|x|\mathbb{1}_{\{|x|\leqslant 10\}}$. Furthermore, if the daily change of the index amounts to $x\%$, the market turnover, in billions of USD, has a uniform distribution over $[5, \frac{1}{2}|x| + 5]$. Let X and Y denote the index change (in percent) and market turnover (in billions of USD) observed on 30.01.2015 r.

   (a) Find $\mathbb{E}(Y|X)$ and $\mathbb{E}Y$.

   (b) Find the joint distribution of the random vector (X, Y) and the marginal distribution of Y.

4. The value of client purchases in a certain kiosk, X_n, are independent random variables from a distribution with a mean 9 of USD and a standard deviation of 1.5 USD. Using the Chebyshev-Bienaymé inequality, assess the probability that the takings on 60 clients will fall into the interval $(525, 555)$ USD. Does the sequence $\frac{X_1+\cdots+X_n}{3n}$ converge almost surely? If yes, find the limit.

5. A brand X car remains failure-free during the warranty period with probability $\frac{3}{4}$. Approximate the probability that among 1200 sold cars, at least 270 and at most 345 will be subject to failure (we assume independence of failures for particular cars). Additionally, if a car fails, the cost of a warranty repair is a random variable with a mean of 2000 and variance equal to 1000^2 (if a car does not break down, the servicing costs are 0). Calculate the expected value of the firm's expenses for warranty service of 1200 cars and approximate the probability that these costs exceed 600000. *Hint. Perhaps not all values from the CLT formula must be calculated in order to provide the answer.*

6. A research center conducts monthly surveys aimed at analyzing the labor market flows between the groups of employed, unemployed and inactive. Available data suggests that in a certain small city, if an individual is inactive in a given period, next month he will remain inactive with probability 0.7 and find employment with probability 0.3. If an individual is employed, then in the next period he will remain employed with probability 0.7, he will lose his job with probability 0.1 and fall out of the labor force with probability 0.2. If, in a given month, an individual is unemployed, then during the next survey he will be found unemployed with probability 0.5, with a job with probability 0.3 and inactive with probability 0.2.

   (a) Assuming that the observed mechanisms have been in effect for a long time, find the average population fraction of those who are inactive.

   (b) Calculate the probability that an individual working in January 2015 will be employed in March 2015.

   (c) An individual found a job in January 2015. Find the expected number of months until he stops working (for any reason), i.e. the average duration of employment.

$\Phi(0) = 0{,}5$, $\Phi(1) \approx 0{,}841$, $\Phi(1{,}5) \approx 0{,}933$, $\Phi(2) \approx 0{,}977$, $\Phi(2{,}5) \approx 0{,}994$, $\Phi(3) \approx 0{,}9987$, $\Phi(4) \approx 0{,}99997$